

Pure Mathematics

2nd secondary

Unit:(1)

Functions of a real variable and drawing curves

- ① Real functions.
- ② Operations on function – composition of functions.
- ③ Some properties of functions (Even, odd – one-to-one functions)
- ④ Graphical representation of basic functions.
- ⑤ Geometric transformations of basic function curves.
- ⑥ Solving absolute value equations and inequalities.

Lesson:(1) Functions of real variable

Definition: The function is a relation between two variables x and y such that for every value of x there exists one and only one value of y .

For examples: $y = x^2 - 1$ is a function because:

When x say 2

$\therefore y = 5$ (one value)

The relations:

1) $y = x^2 + 5x - 3$

3) $y^3 = 5x^2 - 1$

5) $y = 5$

7) $F(x) = \begin{cases} 3x - 2 & x < 1 \\ x^2 - 1 & x \geq 1 \end{cases}$

2) $y = \sqrt{x^2 + 1}$

4) $F(x) = 1 - 2x$

6) $y = \sin x$

Are all functions

But the relations:

1) $y^2 = x^2 + 5x - 3$

3) $x = 7$

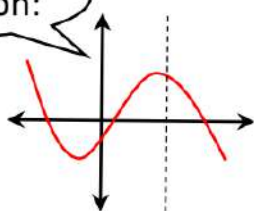
5) $y = \begin{cases} 3x - 2 & x \leq 1 \\ x^2 - 1 & x \geq 1 \end{cases}$

2) $y^4 = 2x^3 - 1$

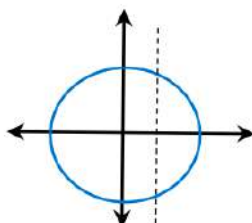
4) $y = \pm\sqrt{x+4}$

Are not functions

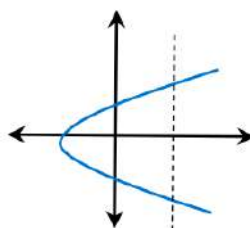
By graph:



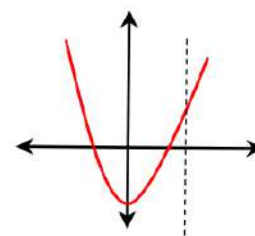
Function



Not Function



Not Function



Function

To graph the curve of any function, we must know:

1) Its domain

2) Its co-domain

3) Its rule

Given:

f : Domain $\longrightarrow$ Co-domain,

$y = f(x)$

Required:

1) graph

2) Domain

3) Range

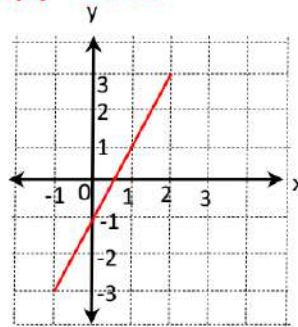
Examples

1) $F: [-1, 2] \longrightarrow \mathbb{R}, F(x) = 2x - 1$

x:	-1	0	2
F(x):	-3	-1	3

Domain = $[-1, 2]$

The range = $[-3, 3]$

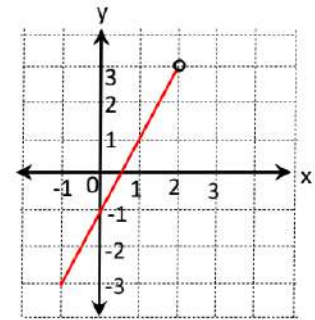


2) $F: [-1, 2[\longrightarrow \mathbb{R}, F(x) = 2x - 1$

x:	-1	0	2
F(x):	-3	-1	3

Domain = $[-1, 2[$

The range = $[-3, 3[$

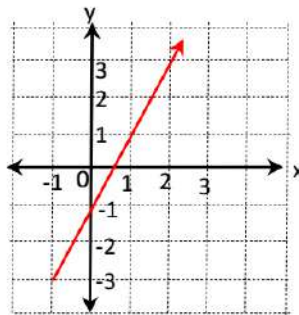


3) $F: [-1, \infty[\longrightarrow \mathbb{R}, F(x) = 2x - 1$

x:	-1	0	2
F(x):	-3	-1	3

Domain = $[-1, \infty[$

The range = $[-3, \infty[$

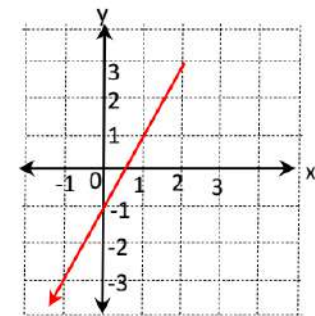


4) $F:]-\infty, 2] \longrightarrow \mathbb{R}, F(x) = 2x - 1$

x:	-1	0	2
F(x):	-3	-1	3

Domain = $]-\infty, 2]$

The range = $]-\infty, 3]$

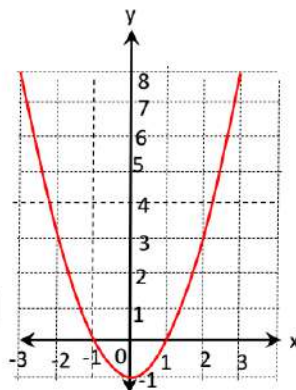


5) $F: [-3, 3] \longrightarrow [-3, 9], F(x) = 2x - 1$

x	-3	-2	-1	0	1	2	3
F(x)	8	3	0	-1	0	3	8

Domain = $[-3, 3]$

The range = $[-1, 8]$

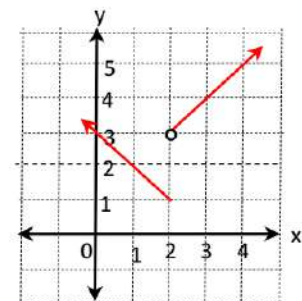


6) $f(x) = \begin{cases} 3 - x, & x \leq 2 \\ x + 1, & x > 2 \end{cases}$

x	0	1	2	2	3	4
f(x)	3	2	1	3	4	5

Domain = $\mathbb{R}$

The range = $[1, \infty[$



The domain algebraically

[1] If $F(x)$ is a polynomial function

$$\therefore D(F) = \mathbb{R}$$

i.e. $F(x) = 13$

$$F(x) = 2x + 1$$

$$F(x) = x^2 + 3$$

$$F(x) = 2x^3 + 2x - 3$$

$$F(x) = \sqrt[3]{x^2 + 9x}$$

$$F(x) = |x + 1|$$

Domain (D) = $\mathbb{R}$

[2] If $F(x)$ is a rational function where $f(x) = \frac{h(x)}{g(x)}$, then $D(F) = \mathbb{R} - \{\text{zeroes of } g(x)\}$

Examples

1) $F(x) = \frac{3}{x}$

$$\therefore D = \mathbb{R} - \{0\}$$

2) $F(x) = \frac{x^2 - 2x}{x}$

$$\therefore D = \mathbb{R} - \{0\}$$

3) $F(x) = \frac{x^2 + 3x + 2}{x - 1}$

$$x - 1 = 0$$

$$\therefore x = 1$$

$$\therefore D = \mathbb{R} - \{1\}$$

4) $F(x) = \frac{5}{x^2 - x - 2}$

$$x^2 - x - 2 = 0$$

$$\therefore (x - 2)(x + 1) = 0 \quad \therefore x = 2, x = -1$$

$$\therefore D = \mathbb{R} - \{2, -1\}$$

5) $F(x) = \frac{x - 1}{x^2 + 4}$

$$x^2 + 4 = 0 \text{ could not be factorized}$$

$\therefore$ there are no real roots in the D_r

$$\therefore D = \mathbb{R}$$

6) $F(x) = \frac{1}{x} + \frac{x}{x^2 - 4}$

$$\therefore D = \mathbb{R} - \{0, 2, -2\}$$

[3] Domain of irrational functions " the n^{th} root "

If $f(x) = \sqrt[n]{g(x)}$ where $n \in \mathbb{Z}^+$, $g(x)$ is a polynomial

First : when (n) is an odd number, then the domain of $f = \mathbb{R}$

Second : when n is an even number, then:

If $\sqrt{\quad} \Rightarrow \begin{cases} \text{Numerator} & \therefore \text{under root} \geq 0 \\ \text{Denominator} & \therefore \text{under root} > 0 \end{cases}$

Examples

1) $F(x) = \sqrt{x-3}$

$x-3 \geq 0 \quad \therefore x \geq 3 \quad \therefore D = [3, \infty[$

2) $F(x) = \frac{2}{\sqrt{x-2}}$

$x-2 > 0 \quad \therefore x > 2 \quad \therefore D =]2, \infty[$

3) $F(x) = \frac{2}{\sqrt{4-x}}$

$4-x > 0 \quad \therefore -x > -4 \quad (\times -1)$
 $x < 4 \quad \therefore D =]-\infty, 4[$

4) 1) $F(x) = 4 - \sqrt{3-2x}$

$3-2x \geq 0 \quad \therefore -2x \geq -3 \quad \therefore x \leq \frac{3}{2}$

$\therefore D =]-\infty, \frac{3}{2}]$

5) $F(x) = \frac{2}{\sqrt{2+3x}}$

$2+3x > 0 \quad \therefore 3x > -2 \quad \therefore x > -\frac{2}{3} \quad \therefore D =]-\frac{2}{3}, \infty[$

6) $F(x) = \sqrt[3]{6-x^2}$

$\therefore$ the index of the root is odd $\therefore D = \mathbb{R}$

7) $F(x) = \sqrt{x^2-4x+4}$

$x^2-4x+4 \geq 0 \quad \therefore (x-2)^2 \geq 0 \quad \therefore D = \mathbb{R}$

8) $F(x) = \frac{1}{\sqrt{x^2-3x-4}}$

$x^2-3x-4 > 0 \quad \therefore (x-4)(x+1) > 0$

$\therefore D = \mathbb{R} -]-1, 4[$



Find the domain of the following functions

$$9) F(x) = \frac{2x-1}{x+2}$$

$$10) F(x) = \frac{x-1}{2x-1}$$

$$11) F(x) = \frac{1}{x^2-3x}$$

$$12) F(x) = \frac{x}{x^2-2x-3}$$

$$13) F(x) = \frac{x}{3x^2-x-2}$$

$$14) F(x) = \frac{x^2-1}{x^3-27}$$

$$15) F(x) = \sqrt{x}$$

$$16) F(x) = \sqrt{x-2}$$

$$17) F(x) = \sqrt{3-x}$$

$$18) F(x) = \sqrt[3]{2x+3}$$

$$19) F(x) = \frac{5}{\sqrt{x}}$$

$$20) F(x) = \frac{5}{\sqrt{x-3}}$$

$$21) F(x) = \sqrt{x^2 - 9}$$

$$22) F(x) = \sqrt{x^2 - x - 6}$$

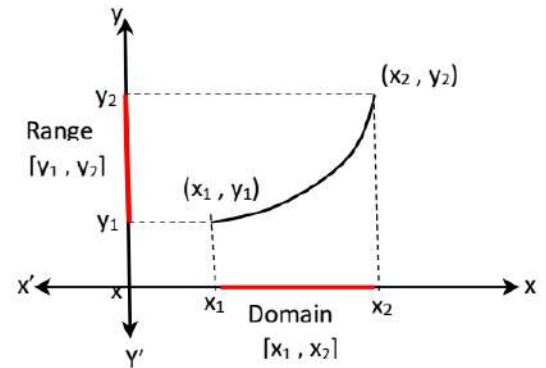
$$23) F(x) = \sqrt{4x^2 - 12x + 9}$$

$$24) F(x) = \frac{1}{\sqrt{2+x-x^2}}$$

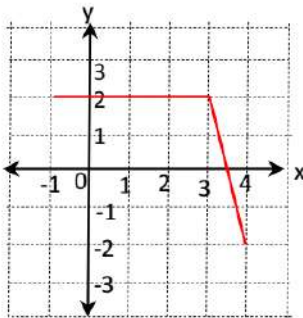
Identify the domain of the function from the graph

(1) Domain of the function is the set of the x-coordinates of all the points lie on the curve of the function

(2) Range of the function is the set of the y-coordinates of all the points lie on the curve of the function

Find the domain and the range of the following functions:

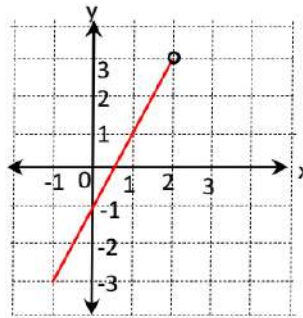
(1)



Domain =

Range =

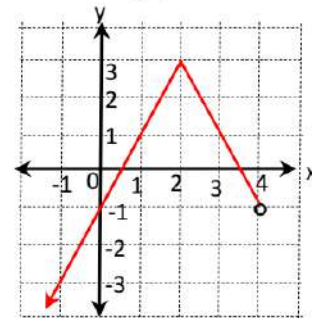
(2)



Domain =

Range =

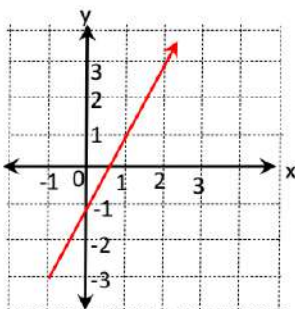
(3)



Domain =

Range =

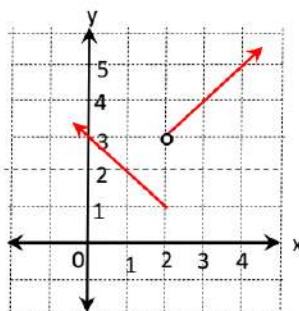
(4)



Domain =

Range =

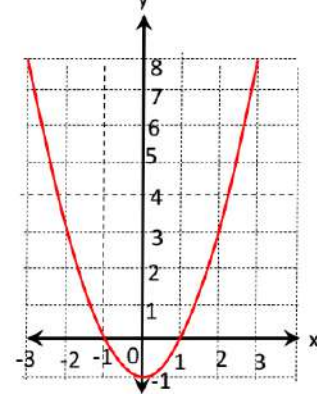
(5)



Domain =

Range =

(6)



Domain =

Range =

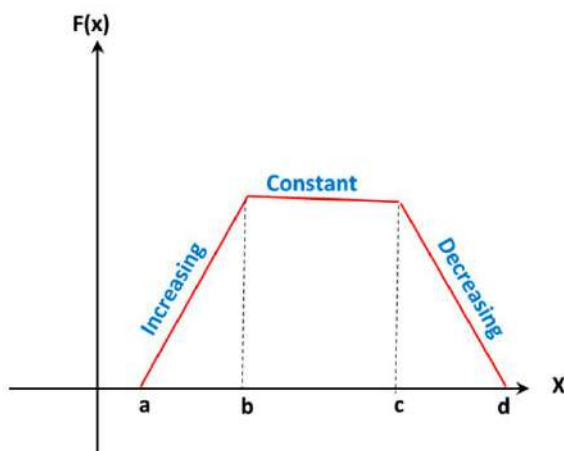
The monotony of the function

In the shown figure:

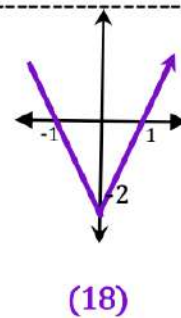
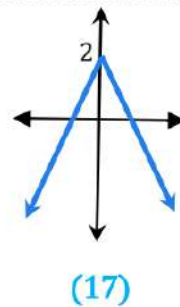
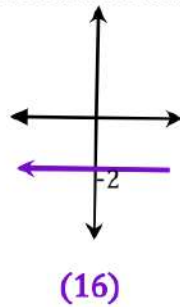
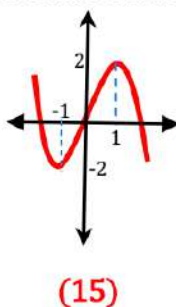
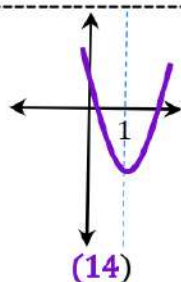
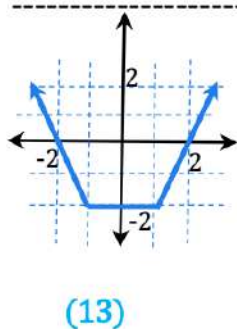
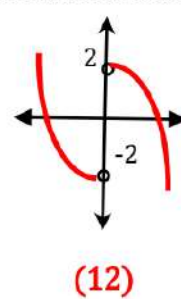
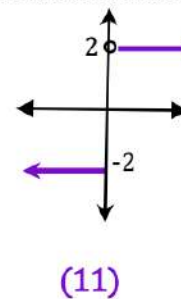
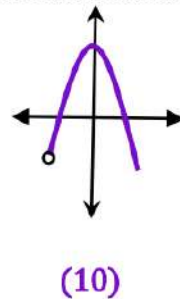
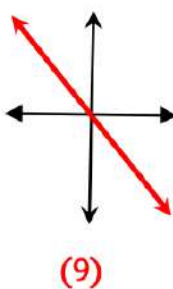
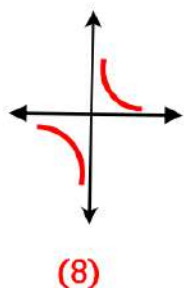
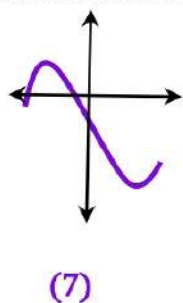
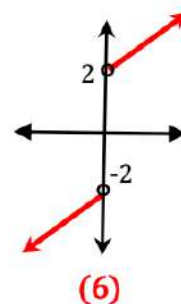
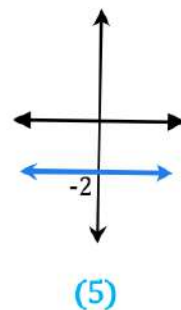
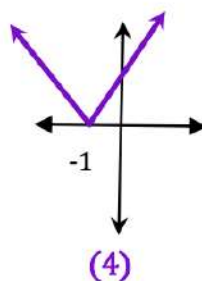
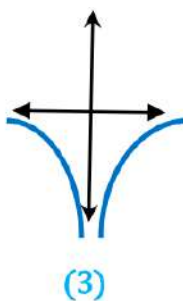
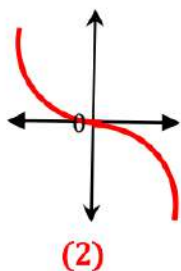
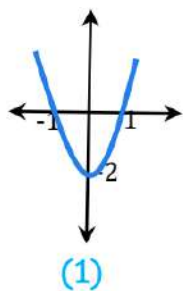
When $x \in [a, b]$ $F(x)$ increasing

When $x \in [b, c]$ $F(x)$ constant

When $x \in [c, d]$ $F(x)$ decreasing



In the following figures, deduce the range, the monotonicity



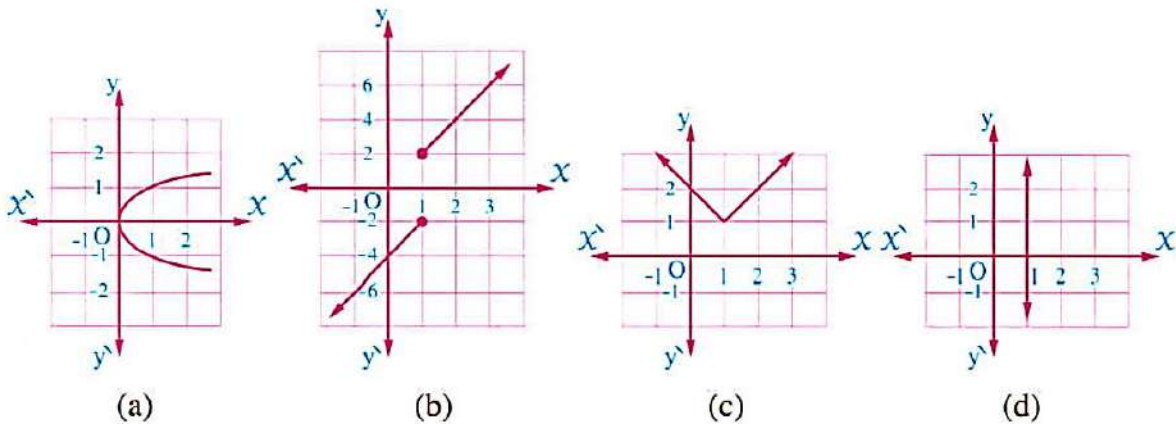
Exercises (1)

Choose the correct answer:

1)	In all the following relations , y is a function in X except	
	(a) $y = 3X + 1$ (b) $y = X^2 - 4$ (c) $X = y^2 - 2$ (d) $y = \sin X$	
2)	In all the following relations , y is a function in X except the relation	
	(a) $y = \cos X$ (b) $y = 2$ (c) $y = X^2 - 1$ (d) $y^2 = X^2 + 1$	
3)	If $f(\sqrt{X}) = X^2$, then $f(2) = \dots\dots\dots$	
	(a) $\sqrt{2}$ (b) 2 (c) 4 (d) 16	
4)	The domain of the function $f : f(X) = 5$ is	
	(a) $\mathbb{R}$ (b) $\mathbb{R}^+$ (c) $\{5\}$ (d) $\{0, 5\}$	
5)	The domain of the function $f : f(X) = \frac{2X+1}{X-2}$ is	
	(a) $\mathbb{R}$ (b) $\mathbb{R} - \{-\frac{1}{2}\}$ (c) $\mathbb{R} - \{-\frac{1}{2}, 2\}$ (d) $\mathbb{R} - \{2\}$	
6)	The domain of the function $f : f(X) = \frac{X^2+1}{X^2+4X}$ is	
	(a) $\mathbb{R} - \{1, -1\}$ (b) $\mathbb{R} - \{0, -4\}$ (c) $\mathbb{R}$ (d) $\mathbb{R} - \{0, 4\}$	
7)	The domain of the function f where $f(X) = \frac{5+2X}{X^2+X+1}$ is	
	(a) $\mathbb{R}$ (b) $\mathbb{R} - \{5\}$ (c) $\mathbb{R} - \{2\}$ (d) $\mathbb{R} - \{-2, -5\}$	
8)	The domain of the function f where $f : \mathbb{R}^+ \longrightarrow \mathbb{R}$, $f(X) = \frac{X-1}{4X}$ is	
	(a) $\mathbb{R}$ (b) $\mathbb{R} - \{0\}$ (c) $\mathbb{R}^+$ (d) $\mathbb{R} - \{1\}$	
9)	 If the domain of the function $f : f(X) = \frac{2}{X^2-6X+k}$ is $\mathbb{R} - \{3\}$, then $k = \dots\dots\dots$	
	(a) 3 (b) 9 (c) ± 9 (d) 18	

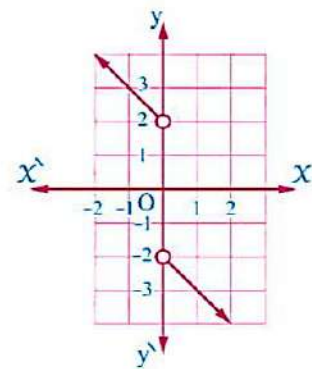
10)	 The domain of the function $f : f(x) = \sqrt{x-3}$ is (a) $\mathbb{R}$ (b) $\mathbb{R} - \{3\}$ (c) $[3, \infty[$ (d) $]-\infty, 3[$	
11)	The domain of the function f where $f(x) = \sqrt{4-x}$ is (a) $[4, \infty[$ (b) $]-\infty, 4[$ (c) $]4, \infty[$ (d) $]-\infty, 4]$	
12)	 The domain of the function $f : f(x) = \sqrt[3]{x-5}$ is (a) $[5, \infty[$ (b) $]-\infty, 5[$ (c) $\mathbb{R}$ (d) $\mathbb{R}^+$	
13)	 The domain of the function $f : f(x) = \frac{5}{\sqrt{x-4}}$ is (a) $[4, \infty[$ (b) $]4, \infty[$ (c) $]-\infty, 4]$ (d) $]-\infty, 4[$	
14)	The domain of the function $f : f(x) = \frac{1}{\sqrt{9-x^2}}$ is (a) $\mathbb{R}$ (b) $\mathbb{R} - [-3, 3]$ (c) $\mathbb{R} - \{-3, 3\}$ (d) $]-3, 3[$	
15)	 The domain of the function f where $f(x) = \frac{1}{\sqrt[3]{x^2-5x-6}}$ is (a) $\mathbb{R} - \{5\}$ (b) $\mathbb{R} - \{6\}$ (c) $\mathbb{R} - \{1, -6\}$ (d) $\mathbb{R} - \{-1, 6\}$	
16)	If the domain of the function $f : f(x) = \frac{1}{\sqrt{x^2+a}}$ is $\mathbb{R} - [-5, 5]$, then $a = \dots\dots\dots$ (a) 5 (b) -25 (c) -5 (d) 25	
17)	The domain of the function f where $f(x) = \begin{cases} -2 & , & x < 2 \\ 3 & , & x > 2 \end{cases}$ is (a) $\mathbb{R}$ (b) $\mathbb{R} - \{3\}$ (c) $\mathbb{R} - \{-2\}$ (d) $\mathbb{R} - \{2\}$	
18)	The range of the function $f : f(x) = \begin{cases} 0 & , & x \leq 0 \\ 1 & , & x > 0 \end{cases}$ is (a) $\{1\}$ (b) $\{0\}$ (c) $\mathbb{R}$ (d) $\{0, 1\}$	

19) The figure which represents y as a function in X is



20) The opposite figure represents function of X , its domain is

- (a) $\mathbb{R}$ (b) $\mathbb{R} -]-2, 2[$
(c) $\mathbb{R} - [-2, 2]$ (d) $\mathbb{R} - \{0\}$



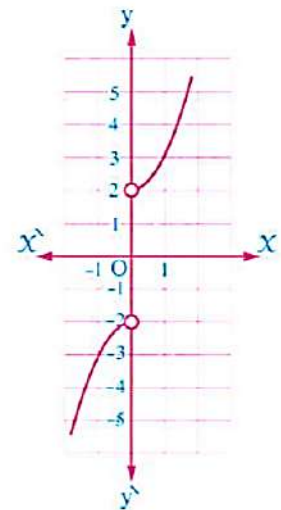
21) In the opposite figure :

First : The range of the function is

- (a) $\mathbb{R} - \{0\}$ (b) $\mathbb{R} - [-2, 2]$
(c) $\mathbb{R}$ (d) $[-2, 2]$

Second : The function is increasing in

- (a) $]-\infty, 0[$ only
(b) $]0, \infty[$ only
(c) $]-\infty, 0[,]0, \infty[$
(d) $\mathbb{R} - [-2, 2]$



Lesson (2): Operations on function – composition of functions.

If F_1, F_2 two functions and their domains are D_1, D_2 respectively, then:

$$[1] (F_1 \pm F_2)(x) = F_1(x) \pm F_2(x) \quad , \quad D(F_1 \pm F_2) = D_1 \cap D_2$$

$$[2] (F_1 \times F_2)(x) = F_1(x) \times F_2(x) \quad , \quad D(F_1 \times F_2) = D_1 \cap D_2$$

$$[3] \left(\frac{F_1(x)}{F_2(x)} \right)(x) = \frac{F_1(x)}{F_2(x)} \text{ Where } F_2(x) \neq 0,$$

$$D\left(\frac{F_1}{F_2}\right) = D_1 \cap D_2 - Z_2 \text{ where } Z_2 \text{ is the set of zeroes of } F_2$$

Examples

[1] If $F: \mathbb{R}^+ \rightarrow \mathbb{R}$ where $F(x) = 2x^2 + 1$,

$G:]-\infty, 4] \rightarrow \mathbb{R}$ where $G(x) = x - 4$. Find:

$$(1) (F + G)(x) \quad (2) (F - G)(x) \quad (3) (F \times G)(x) \quad (4) \frac{F}{G}(x)$$

and find the domain of each, then find: $(F + G)(3), (F - G)(0), (F \times G)(-2), \frac{F}{G}(1)$

Sol.

$$(1) (F + G)(x) = (2x^2 + 1) + (x - 4) \\ = 2x^2 + x - 3$$

$$\text{Domain of } (F + G)(x) = D_1 \cap D_2 \\ = \mathbb{R}^+ \cap]-\infty, 4] =]0, 4]$$

$$(F + G)(3) = 2(3)^2 + (3) - 4 = 17$$

$$(3) (F \times G)(x) = (2x^2 + 1)(x - 4) \\ = 2x^3 - 8x^2 + x - 4$$

$$\text{Domain of } (F + G)(x) = D_1 \cap D_2 \\ = \mathbb{R}^+ \cap]-\infty, 4] =]0, 4]$$

$$(F \times G)(-2) = 2(-2)^3 - 8(-2)^2 + (-2) = -54$$

$$(2) (F - G)(x) = (2x^2 + 1) - (x - 4) \\ = 2x^2 - x + 5$$

$$\text{Domain of } (F + G)(x) = D_1 \cap D_2 \\ = \mathbb{R}^+ \cap]-\infty, 4] =]0, 4]$$

$$(F - G)(0) = 2(0)^2 - (0) + 5 = 5$$

$$(4) \frac{F}{G}(x) = \frac{2x^2 + 1}{x - 4}$$

$$\text{Domain of } \frac{F}{G}(x) = D_1 \cap D_2 - \{Z_2\}$$

Where Z_2 is the set of zeroes of F_2

$$\text{To get } Z_2: x - 4 = 0 \quad x = 4 \quad Z_2 = \{4\}$$

$$\therefore \text{Domain of } \frac{F}{G}(x) = \mathbb{R}^+ \cap]-\infty, 4] - \{4\} \\ =]-\infty, 4[\quad , \quad \frac{F}{G}(1) = \frac{2(1)^2 + 1}{1 - 4} = \frac{3}{-3} = -1$$

[2] If F, G are two function where $F(x) = \sqrt{x - 2}$, $G(x) = \sqrt{4 - x}$, find:

- (1) $(F + G)(x)$ (2) $(F - G)(x)$ (3) $(F \times G)(x)$ (4) $\frac{F}{G}(x)$

And find the domain of each of them.

Sol.

* The composition of functions

[3] If $F(x) = x^2$, $G(x) = x+1$. Find:

(1) $(F \circ G)(1)$

(2) $(G \circ F)(1)$

(3) $(G \circ G)(-1)$

Sol.

$$(1) (F \circ G)(1) = F[G(1)] = F[1+1] = F(2) = (2)^2 = 4$$

$$(2) (G \circ F)(1) = G[F(1)] = G[(1)^2] = G(1) = 1+1 = 2$$

$$(3) (G \circ G)(-1) = G[G(-1)] = G[-1+1] = G(0) = 0 + 1 = 1$$

[4] If $F(x) = 2x + 1$, $G(x) = x^2 - 3$. Find:

(1) $(F \circ G)(x)$

(2) $(G \circ F)(x)$

(3) $(G \circ G)(x)$

Sol.

Note that:

$$[1] (F \circ G)(x) \neq (G \circ F)(x)$$

$$[2] \text{ To determine the domain of } (F \circ G)(x) = \{x: x \in \text{domain of } G, G \in \text{domain of } F\}$$

(i) Find $D_1 = \text{domain of } G$

(ii) Find $D_2 = \text{value of } x \text{ that make } G \text{ in the domain of } F$

(iii) Find $D_1 \cap D_2$ which is the domain of $(F \circ G)(x)$

[5] Find the domain of $(F \circ G)$, if $F(x) = \frac{4}{x-1}$, $G(x) = \frac{3}{x+3}$

Sol.

[6] Find the domain of $(F \circ G)$, if $F(x) = \sqrt{x-2}$, $G(x) = \sqrt{5-x}$

Sol.

Exercises (2)

Choose the correct answer:

1)	If $f(x) = \sqrt[3]{x-1}$, $g(x) = \sqrt{x-1}$, then the domain of $(f-g)$ is (a) $[-1, \infty[$ (b) $[1, \infty[$ (c) $\mathbb{R}$ (d) $]-\infty, 1]$	
2)	The domain of the function $f : f(x) = \sqrt{x-1} + \sqrt{x+2}$ is (a) $[-2, \infty[$ (b) $[1, \infty[$ (c) $[1, \infty[$ (d) $]-2, \infty[$	
3)	The domain of the function $f : f(x) = \sqrt{x+2} - \sqrt{3-x}$ is (a) $\mathbb{R}$ (b) $\mathbb{R} - [-2, 3]$ (c) $[-2, 3]$ (d) $]-2, 3[$	
4)	If $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ where $f(x) = 3x + 1$, and $(f+g)(x) = x^3 + 2x - 1$, then $g(-1) = \dots\dots\dots$ (a) -2 (b) -1 (c) zero (d) 2	

5)	If f is a function where $f = \{(1, 2), (2, 4), (3, 6), (4, 8)\}$ and g is a function where $g = \{(1, 3), (3, 5), (5, 7)\}$, then $f + g = \dots\dots\dots$ (a) $\{(1, 2), (2, 4), (3, 6), (4, 8), (1, 3), (3, 5), (5, 7)\}$ (b) $\{(1, 2), (2, 4), (3, 6), (4, 8), (5, 7)\}$ (c) $\{(1, 5), (3, 11)\}$ (d) $\{(1, 5), (2, 4), (3, 11), (4, 8), (5, 7)\}$
6)	If $f: \mathbb{R}^+ \longrightarrow \mathbb{R}$ where $f(x) = x^2 - 7$, $g: [-3, 2] \longrightarrow \mathbb{R}$ where $g(x) = x^2 + 3$, then $(f - g)(-5) = \dots\dots\dots$ (a) -5 (b) -10 (c) -20 (d) undefined.
7)	The domain of the function $f: f(x) = \sqrt{x-4} \sqrt[3]{x-2}$ is $\dots\dots\dots$ (a) $]4, \infty[$ (b) $[2, \infty[$ (c) $\mathbb{R}$ (d) $[4, \infty[$
8)	If $f: \mathbb{R} \longrightarrow \mathbb{R}$ where $f(x) = x^2 - x$, $g: \mathbb{R} \longrightarrow \mathbb{R}$ where $g(x) = 3x - 2$, then $(f \times g - 6g)(2) = \dots\dots\dots$ (a) 24 (b) 40 (c) 16 (d) -16
9)	If f, g are two real functions where $f(x) = \frac{x-2}{x^2-3x+2}$, $g(x) = x-3$, then $\left(\frac{f}{g}\right)(3) = \dots\dots\dots$ (a) $\frac{1}{2}$ (b) 1 (c) zero (d) undefined.
10)	The domain of the function $f: f(x) = \frac{x-5}{\sqrt{2x-3}}$ is $\dots\dots\dots$ (a) $\mathbb{R} - \{5\}$ (b) $] \frac{3}{2}, \infty[$ (c) $[\frac{3}{2}, \infty[- \{5\}$ (d) $\mathbb{R} - \{\frac{3}{2}\}$
11)	The domain of the function $f: f(x) = \frac{\sqrt{x-2}}{x-3}$ is $\dots\dots\dots$ (a) $\mathbb{R}$ (b) $\{3\}$ (c) $[2, \infty[$ (d) $[2, \infty[- \{3\}$

12)	The domain of the function f where $f(x) = \frac{x+1}{\sqrt[3]{x-1}}$ is (a) $\mathbb{R} - \{1\}$ (b) $\mathbb{R} - \{-1\}$ (c) $[-1, \infty[$ (d) $\mathbb{R}$
13)	The domain of the function f where $f(x) = \frac{x+1}{\sqrt[3]{x-1}}$ is (a) $\mathbb{R} - \{1\}$ (b) $\mathbb{R} - \{-1\}$ (c) $[-1, \infty[$ (d) $\mathbb{R}$
14)	The domain of the function $f : f(x) = \frac{\sqrt{x-4}}{(x-3)(x-5)}$ is (a) $\mathbb{R}$ (b) $\mathbb{R} - \{3, 5\}$ (c) $[4, \infty[$ (d) $[4, \infty[- \{5\}$
15)	If $f(x) = \sqrt{x+5}$, $g(x) = x^2$, then $(f \circ g)(2) = \dots\dots\dots$ (a) 7 (b) 3 (c) 4 (d) 9
16)	If $f(1) = 4$, $g(4) = 7$, then $(g \circ f)(1) = \dots\dots\dots$ (a) 1 (b) 4 (c) 7 (d) 11
17)	If $f : [-2, 3] \longrightarrow \mathbb{R}$ where $f(x) = x + 1$, $g : [0, 2] \longrightarrow \mathbb{R}$ where $g(x) = x^2$, then $(f \circ g)(2) = \dots\dots\dots$ (a) 4 (b) -1 (c) 5 (d) undefined.
18)	If $f(x) = 5x + 4$, $g(x) = 2 - x$, then $(f \circ g)(-2) + (g \circ f)(5) = \dots\dots\dots$ (a) 24 (b) -27 (c) 49 (d) -3
19)	If $f(x) = \sqrt[3]{x}$, $g(x) = x^3$, then $(f \circ g)(x) = \dots\dots\dots$ (a) x (b) x^3 (c) $\sqrt[3]{x}$ (d) x

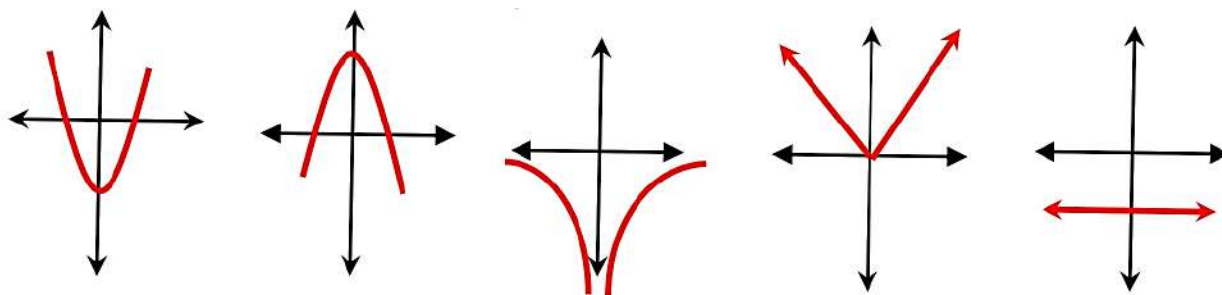
20)	If $f(x) = x - 3$, $g(x) = x^2$, then $(f \circ g)(x) = \dots\dots\dots$ (a) $(x - 3)^2$ (b) $x^2 - 3$ (c) $x^2 + 3$ (d) $\sqrt{x - 3}$
21)	If $f(x) = 2x + 5$ and $g(x) = x^2$, then $f(g(x)) = \dots\dots\dots$ (a) $(2x + 5)^2$ (b) $2x^2 + 5$ (c) $(2x^2 + 5)^2$ (d) $2x^2 + 10$
22)	If $f(x) = ax + 3$, $g(x) = 2x + a$ and $(f \circ g)(2) = 15$, then $a \in \dots\dots\dots$ (a) $\{2, 6\}$ (b) $\{-2, -6\}$ (c) $\{-6, 2\}$ (d) $\{-2, 6\}$
23)	If $f(x) = \frac{1}{x}$, $g(x) = x^2 - 1$, then the domain of $(f \circ g) = \dots\dots\dots$ (a) $\{0, 1, -1\}$ (b) $\mathbb{R} - \{0, 1, -1\}$ (c) $\mathbb{R} - \{1, -1\}$ (d) $[-1, 1]$
24)	If $f(x) = \sqrt{x}$, $g(x) = x^2$, then the domain of $(f \circ g) = \dots\dots\dots$ (a) $]-\infty, 0]$ (b) $[0, \infty[$ (c) $\mathbb{R}^+$ (d) $\mathbb{R}$
25)	The domain of the function $(f \circ g)$ where : $f(x) = \sqrt{x - 2}$, $g(x) = \sqrt{6 - x}$ is $\dots\dots\dots$ (a) $\mathbb{R}$ (b) $[2, \infty[$ (c) $]-\infty, 2]$ (d) $]-\infty, 2[$

Lesson (3): Some properties of functions

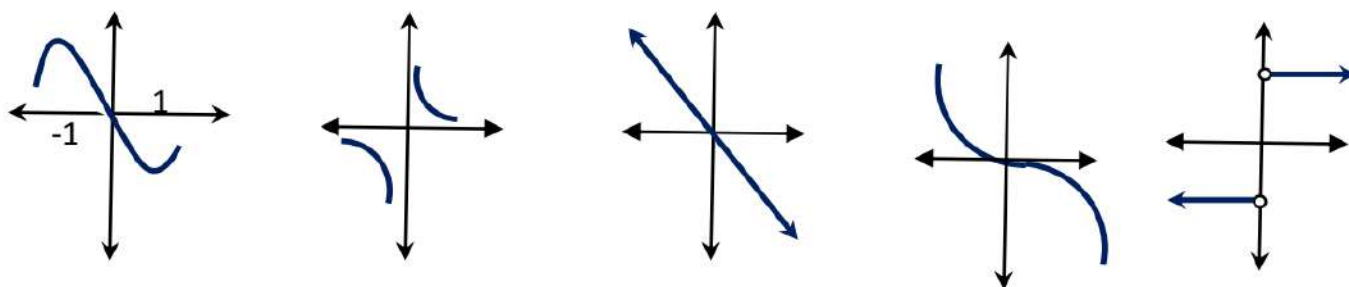
(Even, odd – one-to-one functions)

First: Graphically

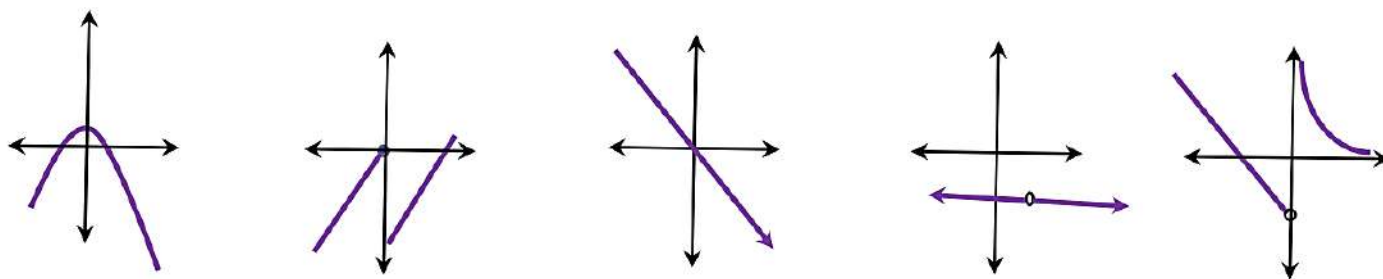
[1] If the curve of the function $F(x)$ is symmetric about Y-axis, then $F(x)$ is even function.



[2] If the curve of the function $F(x)$ is symmetric about the origin point, then $F(x)$ is odd function.



[3] Otherwise, the function is neither even nor odd



Second: Algebraically:

[1] If $F(-x) = F(x)$, then $F(x)$ is even Function

Such as: $F(x) = x^{\text{even}}$, $F(x) = a$ (Constant), $F(x) = \cos x$, $F(x) = |x|$,

[2] If $F(-x) = -F(x)$, then $F(x)$ is odd Function

Such as: $F(x) = x^{\text{odd}}$, $F(x) = \sin x$, $F(x) = \tan x$,

Examples

Which of the following functions is even, odd, or otherwise:

(1) $F(x) = x^2 - 3$

The domain of F is $\mathbb{R}$

$\therefore$ for each x , $-x \in \mathbb{R}$, then

$$F(-x) = (-x)^2 - 3 = x^2 - 3 = F(x)$$

$\therefore F(x)$ is even

(2) $F(x) = \sin x - x^3$

The domain of F is $\mathbb{R}$

$\therefore$ for each x , $-x \in \mathbb{R}$, then

$$F(-x) = \sin(-x) - (-x)^3 = -\sin x + x^3 = -F(x)$$

$\therefore F(x)$ is odd

(3) $F(x) = x^2 - \cos x$

The domain of F is $\mathbb{R}$

$\therefore$ for each x , $-x \in \mathbb{R}$, then

$$F(-x) = (-x)^2 - \cos(-x) = x^2 - \cos x = F(x)$$

$\therefore F(x)$ is even

(4) $F(x) = 5^x - 5^{-x}$

The domain of F is $\mathbb{R}$

$\therefore$ for each x , $-x \in \mathbb{R}$, then

$$F(-x) = 5^{-x} - 5^x = -F(x)$$

$\therefore F(x)$ is odd

(5) $F(x) = x^3 \tan x + \cos x$

for each x , $-x \in \text{domain of } F$, then

$$F(-x) = (-x)^3 \tan(-x) + \cos(-x) = x^3 \tan x + \cos x = F(x)$$

$\therefore F(x)$ is even

(6) $F(x) = x^3 - x^2$

$\therefore$ for each x , $-x \in \mathbb{R}$, then

$$F(-x) = (-x)^3 - (-x)^2 = -x^3 - x^2$$

$\therefore F(x)$ is neither even nor odd (NENO)

(7) $F(x) = \sec x (\sin x + \tan x)$

$$F(-x) = \sec(-x) [\sin(-x) + \tan(-x)]$$

$$= \sec x [-\sin x - \tan x]$$

$$= -\sec x [\sin x + \tan x] = -F(x)$$

$\therefore F(x)$ is odd

(8) $F(x) = \frac{5}{3^x} + 5 \times 3^x$

$$F(x) = \frac{5}{3^{-x}} + 5 \times 3^{-x} = 5 \times 3^x + \frac{5}{3^x}$$

$$= F(x)$$

$\therefore F(x)$ is even

$$(9) F(x) = \frac{\sin x}{x^2 - \tan x}$$

$$F(-x) = \frac{\sin(-x)}{(-x)^2 - \tan(-x)} = \frac{-\sin x}{x^2 + \tan x}$$

∴ F(x) is neither even nor odd (NENO)

$$(11) F(x) = \frac{|x|}{\tan x - \sin x}$$

$$F(-x) = \frac{|-x|}{\tan(-x) - \sin(-x)} = \frac{|x|}{-\tan x - \sin x} = -F(x)$$

∴ F(x) is odd

$$(13) F(x) = \sqrt{x-2}$$

The domain of F is the set of values of x satisfying $x-2 \geq 0$ i.e $x \geq 2$

∴ the domain of F = $[2, \infty[$

∴ $-x \notin [2, \infty[$

∴ F(x) is neither even nor odd (NENO)

$$(15) F(x) = \frac{x^3 \csc 3x}{1+x^4}$$

$$(10) F(x) = \left(x - \frac{1}{x}\right)^3$$

$$F(-x) = \left(-x - \frac{1}{-x}\right)^3 = -\left(x - \frac{1}{x}\right)^3 = -F(x)$$

∴ F(x) is odd

$$(12) f(x) = \left(\frac{x-1}{x+1}\right)^5 - \left(\frac{x+1}{x-1}\right)^5$$

$$F(-x) = \left(\frac{-x-1}{-x+1}\right)^5 - \left(\frac{-x+1}{-x-1}\right)^5 =$$

$$- \left(\frac{x+1}{x-1}\right)^5 - \left(\frac{x-1}{x+1}\right)^5 = -F(x)$$

∴ F(x) is odd

$$(14) F(x) = x^2 + 5, x \in [-3, 3[$$

-3 ∈ the domain of F but 3 ∉ the domain of F

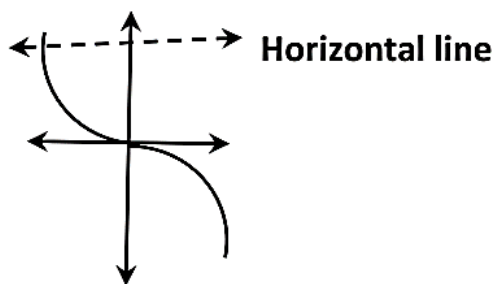
∴ F(x) is neither even nor odd (NENO)

$$(16) F(x) = \frac{3x^3 - \sec x}{x^5 - 4x}$$

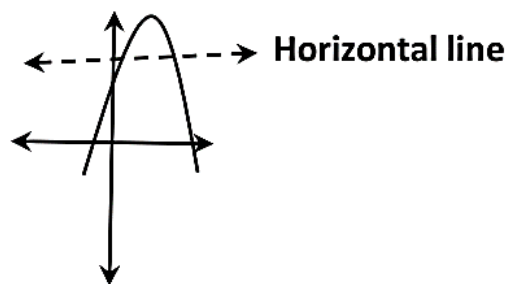
* The one to one function

The function $F: x \longrightarrow y$ is called one to one function if:

for every $a, b \in$ the domain of F and $F(a) = F(b)$, then $a = b$

First: Graphically

The horizontal line cuts the curve in one point, then the function is mono function



The horizontal line cuts the curve in two points, then the function is not mono

Second: Algebraically:Examples

Prove that the following function is one to one function:

[1] $F(x) = 3x - 4$

[2] $F(x) = x^3 - 2$

[3] $F(x) = \frac{4}{3x+1}$

Sol.

[1] Let $a, b \in$ the domain of F

$$\because F(a) = 3a - 4, F(b) = 3b - 4 \text{ and let } F(a) = F(b)$$

$$\therefore 3a - 4 = 3b - 4$$

$$\therefore 3a = 3b$$

$$\therefore a = b$$

$\therefore$ The function F is one to one function.

[2] Let $a, b \in$ the domain of F

$$\because F(a) = a^3 - 2, F(b) = b^3 - 2 \text{ and let } F(a) = F(b)$$

$$\therefore a^3 - 2 = b^3 - 2$$

$$\therefore a^3 = b^3$$

$$\therefore a = b$$

$\therefore$ The function F is one to one function.

[3] Let $a, b \in$ the domain of F

$$\therefore F(a) = \frac{4}{3a+1}, F(b) = \frac{4}{3b+1} \text{ and let } F(a) = F(b)$$

$$\therefore \frac{4}{3a+1} = \frac{4}{3b+1} \quad \therefore 3a+1 = 3b+1 \quad \therefore 3a = 3b \quad \therefore a = b$$

$\therefore$ The function F is one to one function.

Prove that the following two functions is not one to one function:

[4] $F(x) = 2 - x^2$

[5] $F(x) = x^2 + x$

Sol.

Exercises (3)

Choose the correct answer:

1)	The even function from the functions that are defined by the following rules is	
	(a) $f(x) = x^3$ (b) $f(x) = \sin x$	
	(c) $f(x) = x \cos x$ (d) $f(x) = x \sin x$	
2)	The odd function from the functions that are defined by the following rules is	
	(a) $f(x) = x^2 \sin x$ (b) $f(x) = \tan^2 x$	
	(c) $f(x) = \cos x$ (d) $f(x) = 1$	
3)	The type of the function $f : f(x) = \frac{\sin x}{x}$ is	
	(a) even. (b) odd.	
	(c) neither even nor odd. (d) one - to - one.	

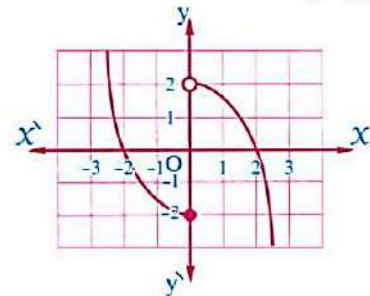
4)	The function $f : f(x) = x \cos x$ is (a) even. (b) odd. (c) neither even nor odd. (d) one - to - one.	
5)	Each of the following is a rule of even function except (a) $f(x) = \sin x$ (b) $f(x) = \cos x$ (c) $f(x) = x^2 - 1$ (d) $f(x) = 1$	
6)	Which of the following rules does not represent an even function ? (a) $y = \frac{1}{x^2}$ (b) $y = \sec x$ (c) $y = x^2 + \sin x$ (d) $y = 3x^4 - 2x^2 + 27$	
7)	If f is an odd function , $f(1) = 2$, then which of the following points lies on the curve of f ? (a) $(-1, 2)$ (b) $(-1, -2)$ (c) $(1, -2)$ (d) $(-1, 0)$	
8)	If f is an odd function , $a \in$ the domain of f , then $f(a) + f(-a) = \dots\dots\dots$ (a) zero. (b) $2f(a)$ (c) $2a$ (d) $f(a)$	
9)	If f is an even function , then $f(a) - f(-a) = \dots\dots\dots$ (a) zero. (b) $f(a)$ (c) $2f(a)$ (d) $f(2a)$	
10)	The function $f : f(x) = (x + \sin x) (\dots\dots\dots)$ is even. (a) $x^3 + \tan x$ (b) $1 + \sin x$ (c) $4 - x^2$ (d) $x - \cos x$	
11)	If f is an even function and the curve of the function passes through point $(-3, 2m + 1)$, and $f(3) = 5$, then $m = \dots\dots\dots$ (a) -1 (b) zero (c) 1 (d) 2	
12)	If the function f is an even over $[a, b]$, then $b = \dots\dots\dots$ (a) a (b) $-a$ (c) $2a$ (d) a^3	
13)	If f is an even function and $f(5) = 1$, $f(-5) = 3 - k$, then $k = \dots\dots\dots$ (a) 1 (b) 5 (c) 3 (d) 2	

14)	The one - to - one function from those defined by the following rules is (a) $f(x) = 3 - x^2$ (b) $f(x) = x^3 + 3$ (c) $f(x) = \sin x \tan x$ (d) $f(x) = x^2 + x$	
15)	The functions defined by the following rules are one - to - one except (a) $f(x) = x^3$ (b) $g(x) = 3x$ (c) $h(x) = \frac{1}{x}$ (d) $n(x) = x^2$	
16)	All functions defined by the following rules are not one - to - one except (a) $f(x) = x^2$ (b) $f(x) = 5$ (c) $f(x) = x + 2$ (d) $f(x) = \sin x$	
17)	If f, g are two functions where $f(x) = x^3$, $g(x) = x + 2$, then $(g \circ f)$ is function. (a) a one - to - one (b) an odd (c) an even (d) a linear	
18)	The function $f : f(x) = x^3 + 5x$ is symmetric about (a) the x-axis. (b) the y-axis. (c) the origin. (d) can not be determined.	
19)	The function $f : f(x) = x^2 + x^4 + 1$ is symmetric about (a) the origin. (b) the x-axis. (c) the y-axis. (d) it has neither symmetric point nor symmetric line.	
20)	The opposite figure represents the curve of the function f, then f is (a) one - to - one. (b) an even function. (c) an odd function. (d) neither odd nor even.	

21)

The opposite figure represents the curve of the function f , then f is

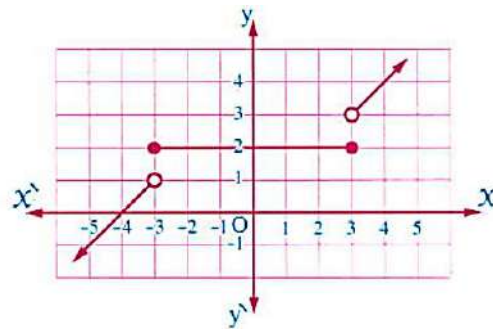
- (a) one - to - one. (b) an even function.
(c) an odd function. (d) neither odd nor even.



22)

The opposite figure represents the curve of the function f , then f is

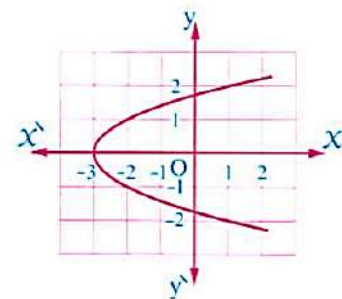
- (a) one - to - one function.
(b) an even function.
(c) an odd function.
(d) neither odd nor even.



23)

 The curve represented in the opposite figure is symmetric about the straight line whose equation is

- (a) $x = 0$ (b) $y = 0$
(c) $y = -2$ (d) $x = 2$



Lesson (4): Graphical representation of functions of basic functions

$$F(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots + a_n x^n$$

is called a polynomial function of n degree,

where n is the highest power of the independent variable x.

where $a_0, a_1, a_2, a_3, \dots, a_n \in \mathbb{R}$, $n \neq 0$, $n \in \mathbb{N}$

Notice:

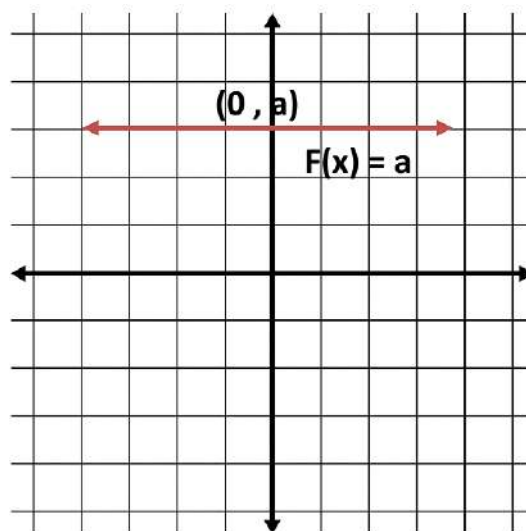
- 1- $F(x) = a_0$, $a_0 \neq 0$ is called a constant polynomial function
- 2- Polynomial functions of first degree are called linear functions, second degree are called quadratic functions, and the third degree are called cubic functions.
- 3- Zeros of the polynomial function are the x-coordinates of the point(s) of the intersection of the curve with the X-axis.
- 4- Two polynomial functions f and g are equal if they have the same degree and the coefficients of corresponding powers of x are equal.

The original functions

First: The constant function:

The function $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = a$, $a \in \mathbb{R}$ is called constant function, represented by a straight line parallel to the x-axis and cuts the y-axis at the point $(0, a)$.

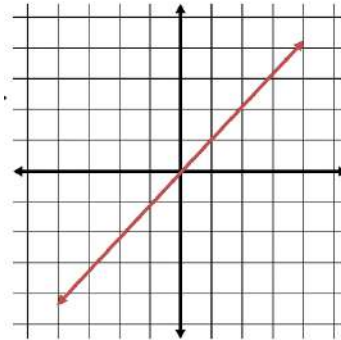
- 1) **Domain** = $\mathbb{R}$
- 2) **Range** = $\{a\}$
- 3) **Mont. : Contant on $\mathbb{R}$**
- 4) Type : even
- 5) Not one-to-one



Second: The linear function:

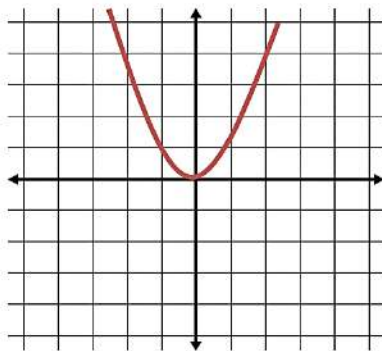
The function $f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = x$ is called a linear function, represented by a st. line passing through the origin point.

- 1) **Domain** = $\mathbb{R}$
- 2) **Range** = $\mathbb{R}$
- 3) **Mont.** :increasing on $\mathbb{R}$
- 4) Type: odd
- 5) one-to-one

**Third: The quadratic function: (Second degree)**

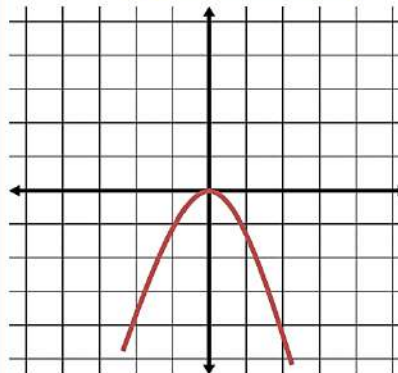
$\therefore f(x) = (x + a)^2 + b, \therefore \text{vertex} = (-a, b)$

$$f(x) = x^2$$



Domain = $\mathbb{R}$
Range = $[0, \infty[$
Type : even
Dec. on $]-\infty, 0[$
Inc. on $]0, \infty[$
Not One - to - one

$$f(x) = -x^2$$

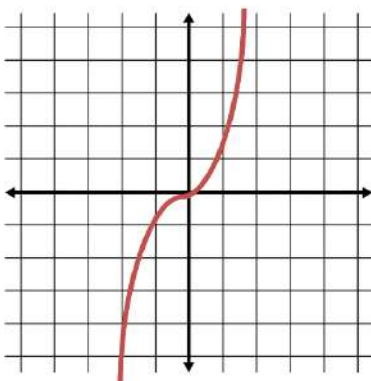


Domain = $\mathbb{R}$
Range = $]-\infty, 0[$
Type : even
Inc. on $]-\infty, 0[$
Dec. on $]0, \infty[$
Not One - to - one

Fourth: The cubic function: (Third degree)

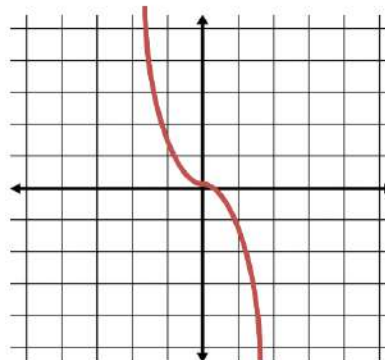
$\therefore f(x) = (x + a)^3 + b, \therefore \text{Point of symmetry} = (-a, b)$

$$f(x) = x^3$$



Domain = $\mathbb{R}$
Range = $\mathbb{R}$
Odd
One - to - one
Inc. on $\mathbb{R}$

$$f(x) = -x^3$$

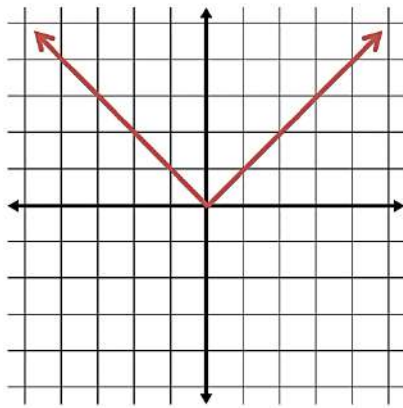


Domain = $\mathbb{R}$
Range = $\mathbb{R}$
Odd
One - to - one
Dec. on $\mathbb{R}$

Fifth: The absolute value function:

$$\therefore f(x) = |x + a| + b, \therefore \text{vertex} = (-a, b)$$

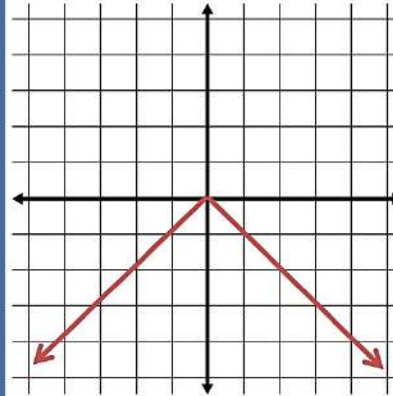
$$f(x) = |x|$$

Domain = $\mathbb{R}$ Range = $[0, \infty[$

Type : even

Dec. on $] -\infty, 0[$ Inc. on $] 0, \infty[$

$$f(x) = -|x|$$

Domain = $\mathbb{R}$ Range = $] -\infty, 0]$

Type : even

Inc. on $] -\infty, 0[$ Dec. on $] 0, \infty[$

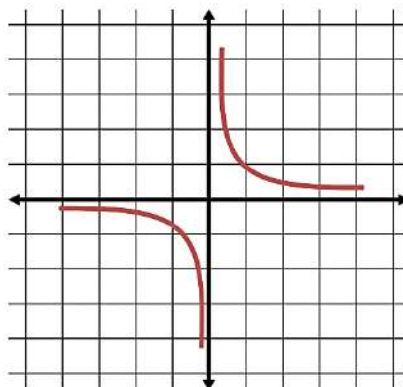
.N

Sixth: The fraction function:

$$\therefore f(x) = \frac{1}{x + a} + b,$$

$$\therefore \text{Point of symmetry} = (-a, b)$$

$$f(x) = \frac{1}{x}$$

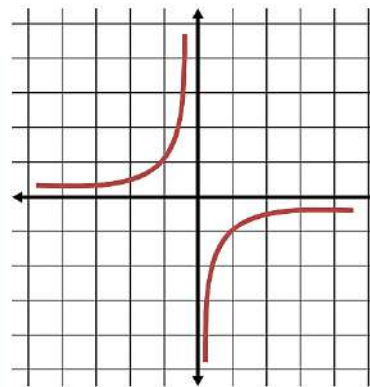
Domain = $\mathbb{R} - \{0\}$ Range = $\mathbb{R} - \{0\}$

Type : Odd

One - to - one

Dec. on its domain

$$f(x) = -\frac{1}{x}$$

Domain = $\mathbb{R} - \{0\}$ Range = $\mathbb{R} - \{0\}$

Type : Odd

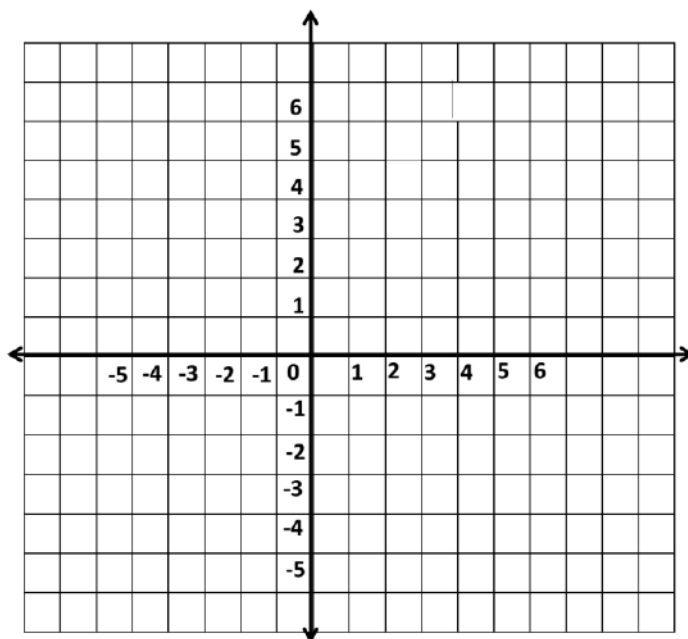
One - to - one

Inc. on its domain

Graph each of the following functions, from the graph find the domain and the range and discuss its monotonicity and its type:

$$(1) \quad f(x) = \begin{cases} x & , x \leq 0 \\ x^2 & , x > 0 \end{cases}$$

Sol.



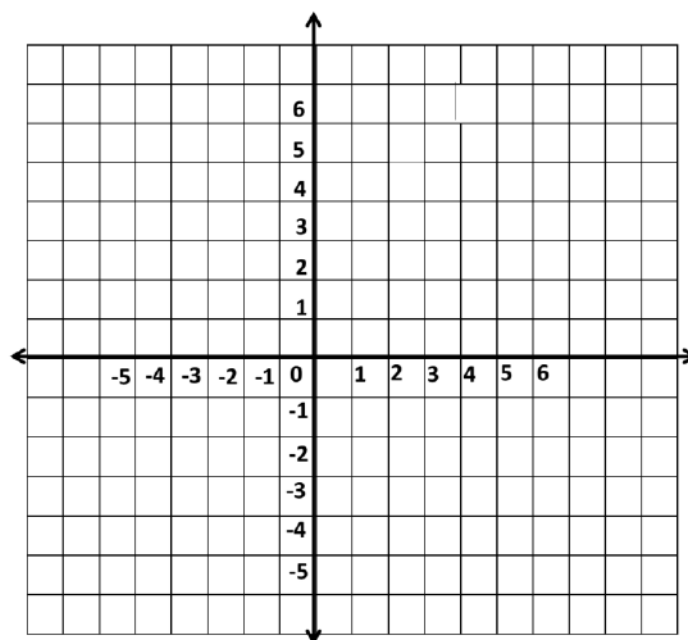
$$(2) \quad f(x) = \begin{cases} |x| & , x \leq 0 \\ \frac{1}{x} & , x > 0 \end{cases}$$

Sol.

$$f(x) = \begin{cases} -x & , x \leq 0 \\ \frac{1}{x} & , x > 0 \end{cases}$$

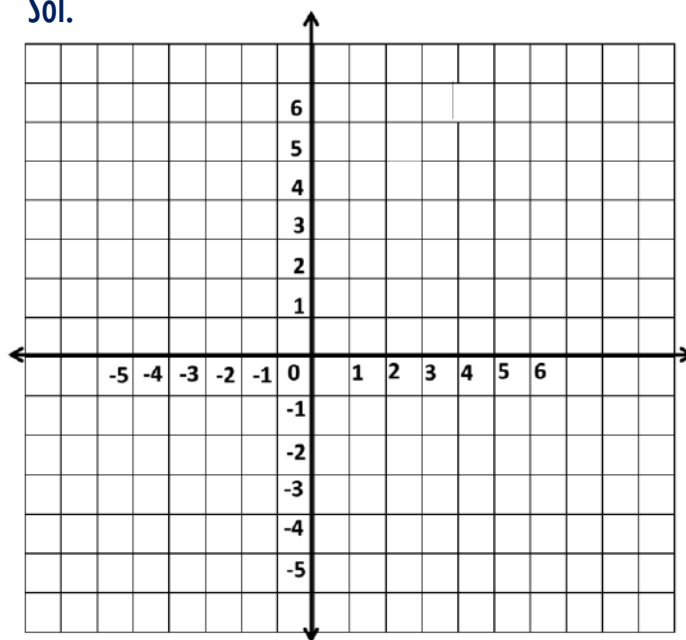
$$f(x) = -x, x \leq 0$$

$$f(x) = \frac{1}{x}, x > 0$$



$$(3) \quad f(x) = \begin{cases} x^3 & , x < 1 \\ 1 & , x > 1 \end{cases}$$

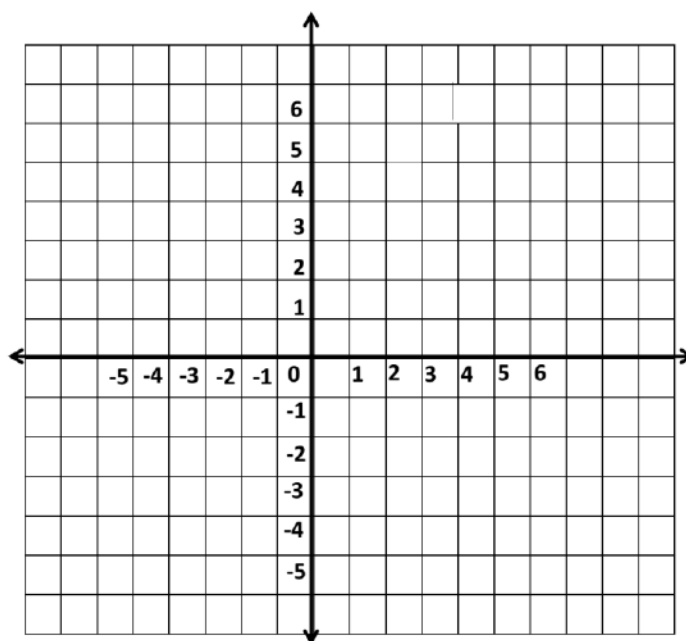
Sol.



$$(4) \quad f(x) = \frac{4 - x^2}{x + 2} \quad \text{Sol.}$$

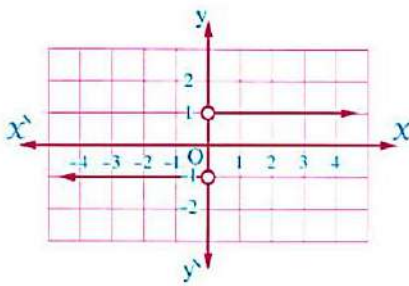
$$f(x) = \frac{-(x-2)(x+2)}{x+2} = -x + 2$$

x	-4	-3	-2	-1	0
y	6	5	4	3	2



Exercises (4)

Choose the correct answer:

1)	If $f(x) = 5$, then the domain of the function f is (a) $\mathbb{R}$ (b) $\mathbb{R}^+$ (c) $\{5\}$ (d) $\mathbb{R} - \{5\}$	
2)	If $f(x) = 7$, then the range of the function f is (a) $\mathbb{R}$ (b) $\mathbb{R}^+$ (c) $\{7\}$ (d) $\mathbb{R} - \{7\}$	
3)	The range of the function $f : f(x) = \begin{cases} 0 & \text{when } x \leq 0 \\ 1 & \text{when } x > 0 \end{cases}$ is (a) $\{1\}$ (b) $\{0\}$ (c) $\mathbb{R}$ (d) $\{0, 1\}$	
4)	In the opposite figure : The range of the function is (a) $\{1\}$ (b) $\{1, -1\}$ (c) $\{-1\}$ (d) $\mathbb{R}$ 	
5)	The range of the function $f : f(x) = \frac{3x^2 - 3}{x^2 - 1}$ is (a) $\mathbb{R} - \{1, -1\}$ (b) $\mathbb{R} - \{3, -3\}$ (c) $\{3, -3\}$ (d) $\{3\}$	
6)	The range of the function $f : f(x) = \frac{2x^3 - 2x}{x^2 - 1}$ is (a) $\mathbb{R} - \{1, -1\}$ (b) $\mathbb{R} - \{2, -2\}$ (c) $\mathbb{R}^+$ (d) $\mathbb{R}^+ - \{2\}$	
7)	The range of the function $f : f(x) = \begin{cases} x & , x > 0 \\ -2 & , x \leq 0 \end{cases}$ is (a) $\mathbb{R}^+$ (b) $\mathbb{R}^+ - \{-2\}$ (c) $\mathbb{R}^+ \cup \{-2\}$ (d) $\mathbb{R}$	

8)

 The function f where $f(x) = \begin{cases} 2 & , \quad x > 0 \\ -2 & , \quad x < 0 \end{cases}$ is symmetric about the point

(a) (2 , 0)

(b) (-2 , 0)

(c) (0 , 0)

(d) (2 , -2)

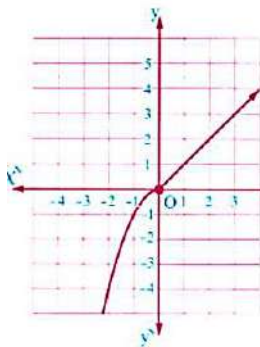
9)

The function $f : f(x) = \begin{cases} -x^2 & , \quad x < 0 \\ \frac{1}{x} & , \quad x > 0 \end{cases}$ is increasing on

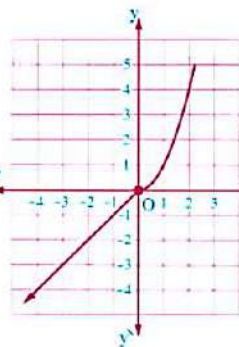
(a) $\mathbb{R}$ (b) $\mathbb{R}^-$ (c) $\mathbb{R}^+$ (d) $\mathbb{R} - \{0\}$

10)

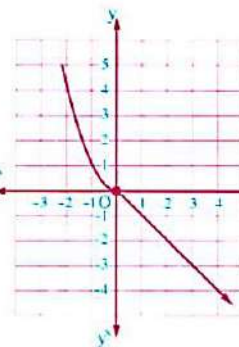
The curve of the function $f : f(x) = \begin{cases} x^2 & , \quad x > 0 \\ x & , \quad x \leq 0 \end{cases}$ is



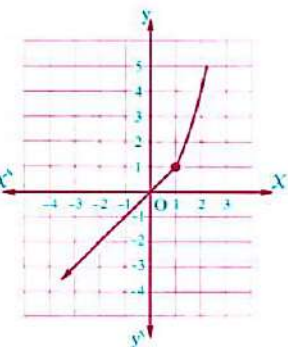
(a)



(b)



(c)



(d)

11)

In the opposite figure :

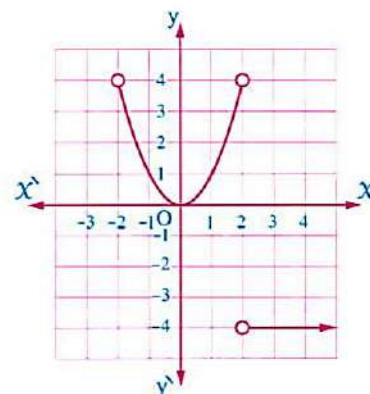
The curve of the function f defined by the rule $f(x) = \dots\dots\dots$

(a) $\begin{cases} x^2 & , \quad -2 < x < 2 \\ -4 & , \quad x < -2 \end{cases}$

(b) $\begin{cases} x^2 & , \quad -2 < x < 2 \\ -4 & , \quad x > 2 \end{cases}$

(c) $\begin{cases} x^2 & , \quad -2 \leq x \leq 2 \\ -4 & , \quad x < 2 \end{cases}$

(d) $\begin{cases} x^2 & , \quad -2 < x < 2 \\ -4 & , \quad x < 2 \end{cases}$



Lesson (5): Graphical transformations of functions of basic functions

To draw any given curve, you must get two elements:

- (i) The vertex point or the point of symmetry (P.O.S)
- (ii) The sign of the function

For example:

- (a) $F(x) = |x + 2| + 1$ $\therefore$ the vertex = $(-2, 1)$ +ve sign
- (b) $F(x) = (x - 1)^2$ $\therefore$ the vertex = $(1, 0)$ +ve sign
- (c) $F(x) = 3 - x^3$ $\therefore$ (P.O.S) = $(0, 3)$ -ve sign
- (d) $F(x) = |x| - 1$ $\therefore$ the vertex = $(0, 1)$ +ve sign
- (e) $F(x) = 2 - (x - 1)^2$ $\therefore$ the vertex = $(1, 2)$ -ve sign
- (f) $F(x) = \frac{1}{x-2}$ $\therefore$ (P.O.S) = $(2, 0)$ +ve sign

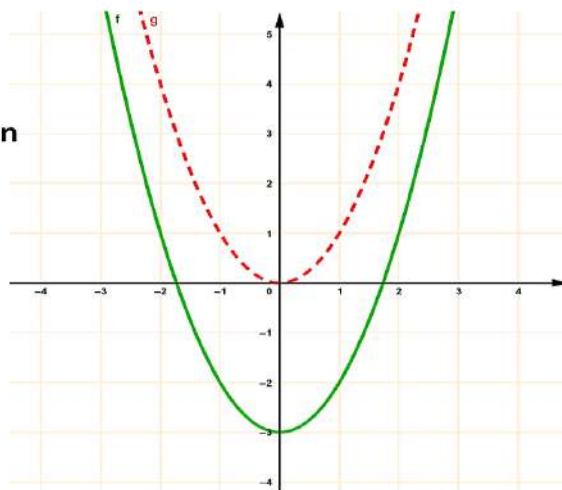


Use the curve of the function f where $g(x) = x^2$ to graph the curve of the function, then deduce its domain and its range and discuss its monotonicity and its type and write equation of axis of symmetry:

(1) $f(x) = x^2 - 3$

Sol.

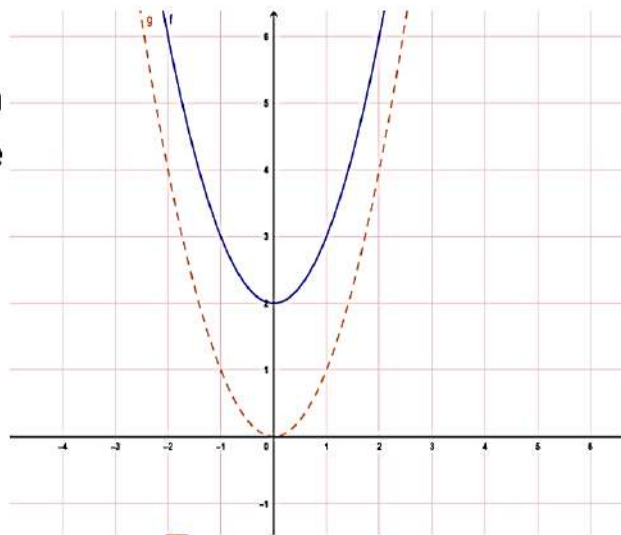
- Point of symmetry = $(0, -3)$
- Is the same curve of the function $g(x) = x^2$ by translation 3 units in the $-ve$ direction of y - axis
- Domain = R
- Range = $[-3, \infty[$
- Dec. = $] -\infty, 0[$
- Inc. = $] 0, \infty[$
- Even
- Equation : $x = 0$



(2) $f(x) = x^2 + 2$

Sol.

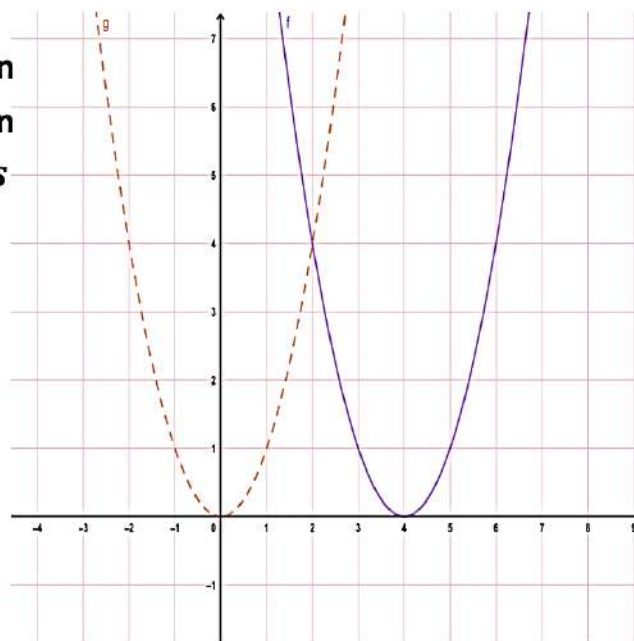
- Point of symmetry = $(0, 2)$
- Is the same curve of the function $g(x) = x^2$ translation 2 units in the $+ve$ direction of y - axis
- Domain = R
- Range = $[2, \infty[$
- Dec. = $] -\infty, 0[$
- Inc. = $] 0, \infty[$
- Even , Equation : $x = 0$

**Second: The horizontal translation:**

(3) $f(x) = (x - 4)^2$

Sol.

- Point of symmetry = $(4, 0)$
- Is the same curve of the function $g(x) = x^2$ by horizontal translation 4 units in $+ve$ direction of x - axis
- Domain = R
- Range = $[0, \infty[$
- Dec. = $] -\infty, 4[$
- Inc. = $] 4, \infty[$
- N.N
- Equation : $x = 4$

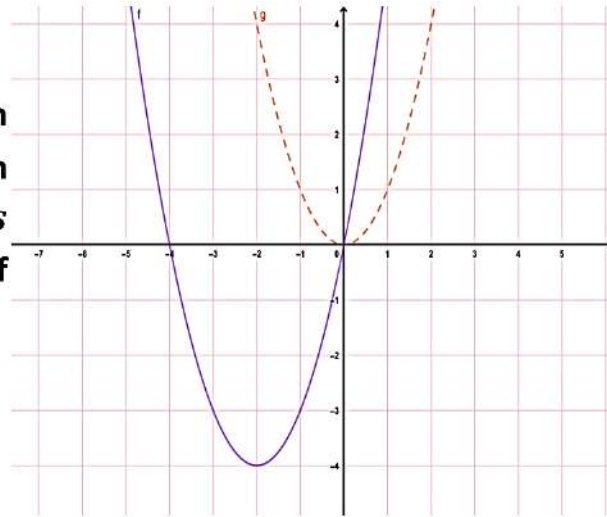


Third: The horizontal and vertical translation:

(4) $f(x) = (x + 2)^2 - 4$

Sol.

- Point of symmetry = $(-2, -4)$
- Is the same curve of the function $g(x) = x^2$ by horizontal translation 2 units in $-ve$ direction of x -axis and 4 units in the $-ve$ direction of y -axis
- Domain = R
- Range = $[-4, \infty[$
- Dec. = $]-\infty, 2[$
- Inc. = $]2, \infty[$
- N.N , Equation : $x = -2$



(5) $f(x) = x^2 - 4x + 5$

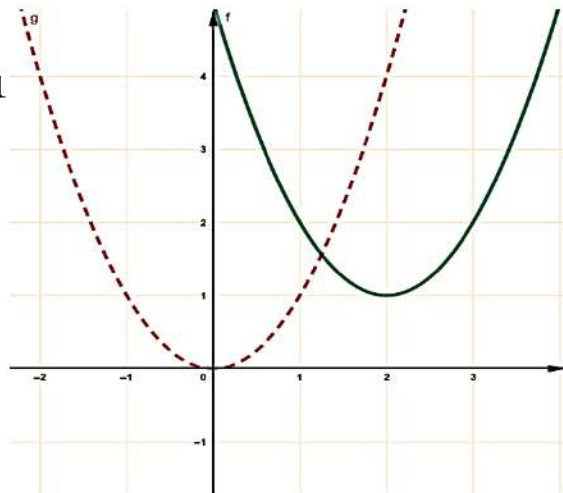
Sol.

Put the fn $f(x) = (x + a)^2 + b$ by completing $(x^2 - 4x)$ to be a perfect square

$$3^{\text{rd}} \text{ term} = \frac{(\text{Middle term})^2}{4 \times 1^{\text{st}}} = \frac{(-4x)^2}{4 \times x^2} = \frac{16x^2}{4x^2} = 4$$

$$\therefore f(x) = (x^2 - 4x + 4) + (-4 + 5) = (x - 2)^2 + 1$$

- Is the same curve of the function $g(x) = x^2$ by translation 2 units in the direction of $\overrightarrow{OX}$ and 1 unit in the direction of $\overrightarrow{OY}$
- Point of symmetry = $(2, 1)$
- Domain = R
- Range = $[1, \infty[$
- Dec. = $]-\infty, 2[$
- Inc. = $]2, \infty[$
- N.N



Use the curves of the standard functions to represent the curves of the functions:

(1) $F(x) = x - 2$ Domain = Range = Type Monotony One – to – one (yes or no)	(2) $F(x) = x + 1 - 2$ Domain = Range = Type Monotony One – to – one (yes or no)
(3) $F(x) = (x - 2)^2 - 1$ Domain = Range = Type One – to – one (yes or no) Increasing on Decreasing on	(4) $F(x) = \frac{2x^2 - 2}{x^2 - 1}$ Domain = Range = Type One – to – one (yes or no) Constant on
(5) $F(x) = \frac{x^2 + x}{x + 1}$ Domain = Range = Type Monotony One – to – one (yes or no)	(6) $F(x) = x^2 - 4x + 4$ Domain = Range = Type Monotony One – to – one (yes or no)

Use the curves of the standard functions to represent the curves of the functions:

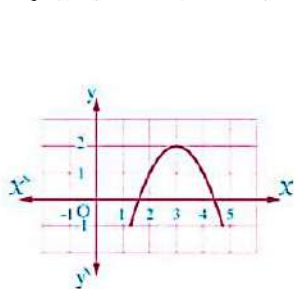
(1) $F(x) = -(x - 1)^3$	(2) $F(x) = 1 - x - 2$
Domain =	Domain =
Range =	Range =
Type	Type
Monotony	Monotony
One – to – one (yes or no)	One – to – one (yes or no)
(3) $F(x) = \frac{1}{2-x} + 1$	(4) $F(x) = -x^2 + 4x - 3$
Domain =	Domain =
Range =	Range =
Type	Type
One – to – one (yes or no)	Constant on
Increasing on Decreasing on	One – to – one (yes or no)
(5) $F(x) = 4 - x^2$	(6) $F(x) = 4 - x^2$
Domain =	Domain =
Range =	Range =
Type	Type
Monotony	Monotony
One – to – one (yes or no)	One – to – one (yes or no)

Exercises (5)

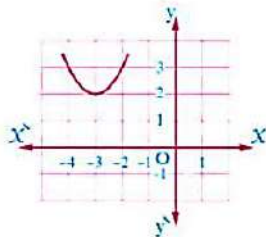
Choose the correct answer:

1)

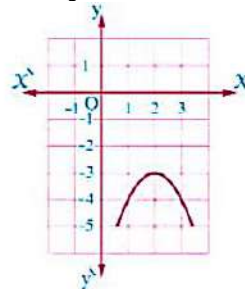
If $f(x) = -(x-3)^2 + 2$, then the graph that represents the function f is



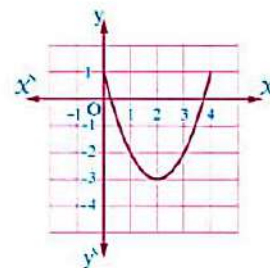
(a)



(b)



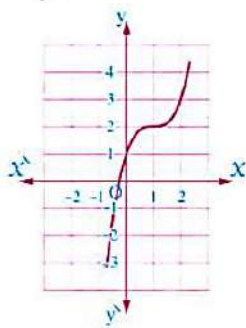
(c)



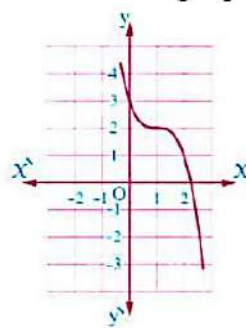
(d)

2)

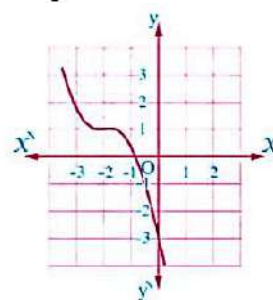
If $f(x) = 2 - (x-1)^3$, then the graph that represents the function f is



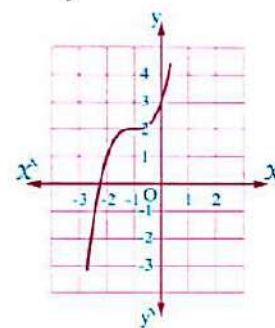
(a)



(b)



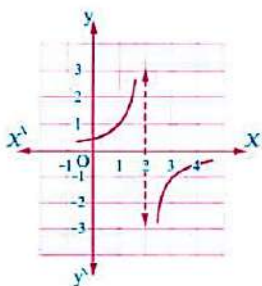
(c)



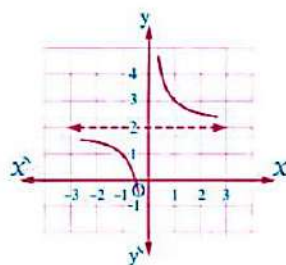
(d)

3)

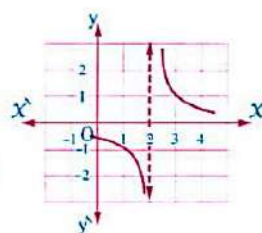
If $f(x) = \frac{1}{x-2}$, then the graph that represents the function f is



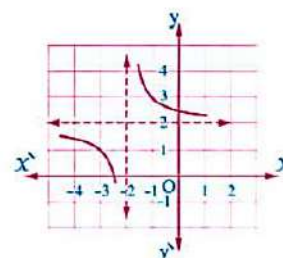
(a)



(b)



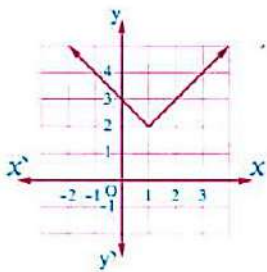
(c)



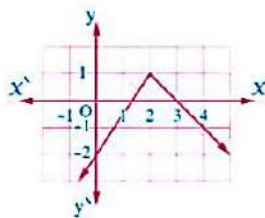
(d)

4)

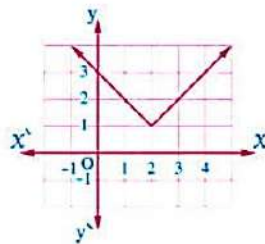
If $f : f(x) = 1 - |x - 2|$ then the figure which represents the function f is



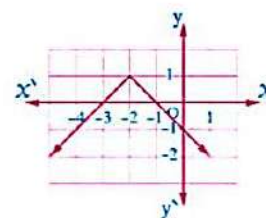
(a)



(b)



(c)



(d)

5)

If the curve of the function $g : g(x) = x^2$ is translated two units in the positive directions of the two axes then the function represents this translation is f :

(a) $f(x) = (x + 2)^2 + 2$

(b) $f(x) = (x + 2)^2 - 2$

(c) $f(x) = (x - 2)^2 - 2$

(d) $f(x) = (x - 2)^2 + 2$

6)

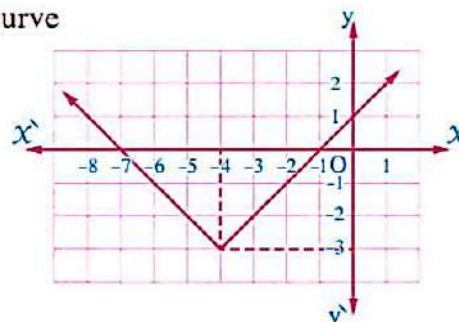
Which of the following function rules represents the curve in the given figure ?

(a) $f(x) = |x - 4| - 3$

(b) $f(x) = |x - 4| + 3$

(c) $f(x) = |x + 4| - 3$

(d) $f(x) = |x + 4| + 3$



7)

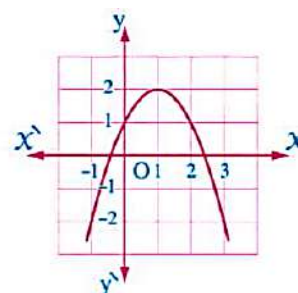
Which of the following function rules is represented in the given figure ?

(a) $f(x) = (x - 1)^2 + 2$

(b) $f(x) = 1 - (x - 2)^2$

(c) $f(x) = 2 - (x - 1)^2$

(d) $f(x) = (x + 1)^2 - 2$



8)

 The point of the vertex of the curve of the function $f : f(x) = (2 - x)^2 + 3$ is

(a) (2, 3)

(b) (2, -3)

(c) (-2, 3)

(d) (-2, -3)

9)	The symmetric point of the function $f : f(x) = 3 - (x + 2)^2$ is	(a) (3 , 2)	(b) (2 , 3)	(c) (-2 , 3)	(d) (-2 , -3)
10)	 The point of symmetry of the curve of the function $f : f(x) = \frac{1}{x-3} + 4$ is	(a) (3 , -4)	(b) (-3 , -4)	(c) (3 , 4)	(d) (-3 , 4)
11)	The symmetric point of the function $f : f(x) = \frac{x+1}{x}$ is	(a) (1 , 0)	(b) (0 , 1)	(c) (0 , 0)	(d) (1 , -1)
12)	 If f is a function where $f(x) = \frac{1}{x}$, then the symmetric point of the function $g : g(x) = f(x + 1)$ is	(a) (1 , 0)	(b) (0 , 1)	(c) (-1 , 0)	(d) (-1 , 1)
13)	The vertex of the curve of the function $f : f(x) = x + 3 - 2$ is	(a) (3 , 2)	(b) (-3 , -2)	(c) (-3 , 2)	(d) (3 , -2)
14)	The curve of the function $f : f(x) = x - 2 $ is symmetric about the straight line	(a) $x = 2$	(b) $x = -2$	(c) $y = 2$	(d) $y = -2$
15)	The range of the function $f : f(x) = \frac{1}{ x }$ is	(a) $\mathbb{R} - \{0\}$	(b) $]0 , \infty[$	(c) $[0 , \infty[$	(d) $\{0\}$
16)	The range of the function $f : f(x) = \frac{2x^2 - 2x}{x - 1}$ equals	(a) $\mathbb{R} - \{1\}$	(b) $\mathbb{R} - \{2\}$	(c) $\mathbb{R} - \{1 , 2\}$	(d) $\mathbb{R} - \{0 , 1\}$
17)	The range of the function $f : f(x) = 2 - 3 - 2x $ is	(a) $] - \infty , 2]$	(b) $[-2 , \infty[$	(c) $] \frac{3}{2} , \infty[$	(d) $] - \infty , -2]$

18)	If $y = f(x)$ is a real function , then its image by translation 3 units vertically upwards is $g(x) = \dots\dots\dots$ (a) $f(x - 3)$ (b) $f(x + 3)$ (c) $f(x) + 3$ (d) $(x) - 3$
19)	 The curve of the function $g : g(x) = x^2 + 4$ is the same curve of the function $f : f(x) = x^2$ by a translation of magnitude 4 units in the direction of $\dots\dots\dots$ (a) $\overrightarrow{OX}$ (b) $\overrightarrow{OX}$ (c) $\overrightarrow{Oy}$ (d) $\overrightarrow{Oy}$
20)	The curve of the function g where $g(x) = x - 2$ is the same as the curve of the function $f : f(x) = x$ by translation two units in direction of $\dots\dots\dots$ (a) $\overrightarrow{OX}$ (b) $\overrightarrow{OX}$ (c) $\overrightarrow{Oy}$ (d) $\overrightarrow{Oy}$
21)	 The curve of the function $g : g(x) = x + 3$ is the same curve of the function $f : f(x) = x$ by a translation of magnitude 3 units in the direction of $\dots\dots\dots$ (a) $\overrightarrow{OX}$ (b) $\overrightarrow{OX}$ (c) $\overrightarrow{Oy}$ (d) $\overrightarrow{Oy}$
22)	If $y = f(x)$ is a real function , then its image by translation 4 units to the left is $g(x) = \dots\dots\dots$ (a) $f(x - 4)$ (b) $f(x + 4)$ (c) $f(x) + 4$ (d) $f(x) - 4$

Lesson (6): Solving absolute value equations and inequalities

[1] Solve the equation: $|x - 1| = 3$ graphically and algebraically.

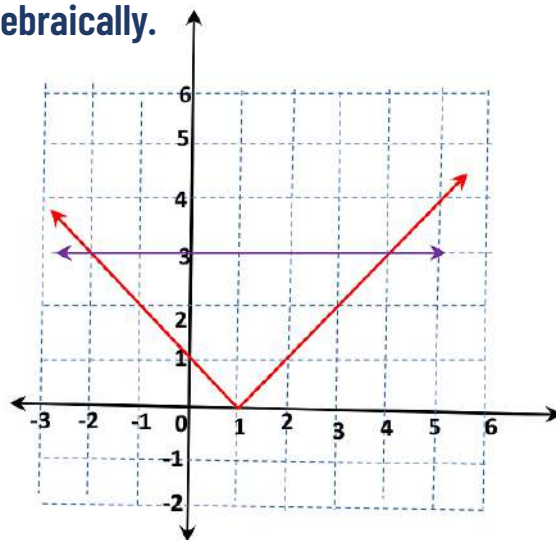
Graphically:

Let $f(x) = |x - 1|$, $g(x) = 3$

Graph the two functions $f(x)$ and $g(x)$ and get

The intersection points, which are $(-2, 3)$ and $(4, 3)$

$\therefore$ The solution set of the equation = $\{-2, 4\}$



Algebraically:

$$F(x) = |x - 1| = \begin{cases} x - 1, & x \geq 1 \\ -x + 1, & x < 1 \end{cases}$$

$x \geq 1$		$x < 1$
$x - 1 = 3$		$-x + 1 = 3$
$x = 4$		$x = -2$

$\therefore$ The solution set of the equation = $\{-2, 4\}$

Properties of the absolute value

(1) $|ab| = |a||b|$

(2) $|a + b| \leq |a| + |b|$

(3) If $|x| = a$, then $x = \pm a$

(4) If $|a| = |b|$, then $a = \pm b$

(5) $|x|^2 = |x^2| = x^2$

(6) $\sqrt{x^2} = |x|$

[2] Solve the equation: $|2x - 1| = x + 2$ graphically and algebraically.

$$F(x) = |2x - 1| = \begin{cases} 2x - 1, & x \geq \frac{1}{2} \\ -2x + 1, & x < \frac{1}{2} \end{cases}$$

$$x \geq \frac{1}{2} \qquad x < \frac{1}{2}$$

$$2x - 1 = x + 2$$

$$-2x + 1 = x + 2$$

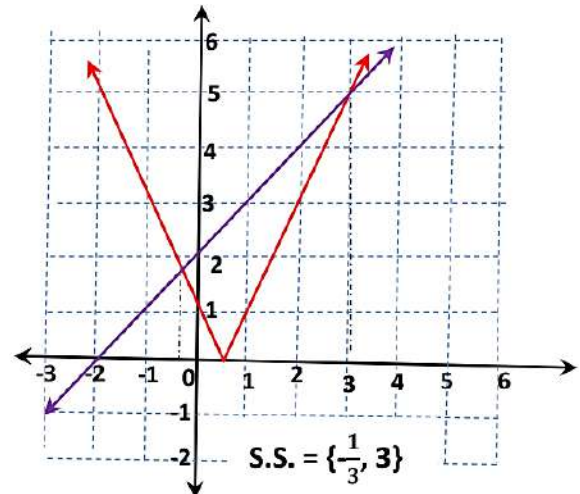
$$2x - x = 2 + 1$$

$$-2x - x = 2 - 1$$

$$x = 3$$

$$-3x = 1 \quad \therefore x = -\frac{1}{3}$$

$\therefore$ The solution set of the equation = $\{-\frac{1}{3}, 3\}$



[3] Find the solution set for each algebraically:

(1) $|x + 7| = |x - 5|$

(2) $\sqrt{x^2 + 6x + 9} = 9 - 2x$

Sol.

(1) $\therefore |x + 7| = |x - 5|$

$\therefore x + 7 = \pm (x - 5)$

$\therefore x + 7 = x - 5$

Or $x + 7 = -x + 5$

$\therefore 2x = 2$

$7 = -5$ (refused)

$\therefore x = -1$

$\therefore$ Solution set = $\{-1\}$

Remember that:

If $a, b \in \mathbb{R}$, and $|a| = |b|$

Then $a = \pm b$

(2) $\therefore \sqrt{x^2 - 6x + 9} = 9 - 2x$

$\therefore \sqrt{(x - 3)^2} = 9 - 2x$

$\therefore |x - 3| = 9 - 2x$

First: when $x \geq 3$

Second: when $x < 3$

$x - 3 = 9 - 2x$

$x - 3 = -9 + 2x$

$3x = 12 \quad \therefore x = 4$ ($\checkmark$)

$x = 6 < 3$ (refused)

$\therefore$ Solution set = $\{4\}$

Remember that:

For any real number a

$\sqrt{a^2} = |a|$

Another solution:

$\therefore \sqrt{x^2 - 6x + 9} = 9 - 2x$

(S.B.S)

$x^2 - 6x + 9 = (9 - 2x)^2$

$\therefore x^2 - 6x + 9 = 81 - 36x + 4x^2$

$\therefore 3x^2 - 30x - 72 = 0$

($\div 3$)

$\therefore x^2 - 10x - 24 = 0$

$\therefore (x - 4)(x - 6) = 0 \quad \therefore x = 4 \text{ or } x = 6$

By substitution:

When $x = 4$

L.H.S. = $\sqrt{4^2 - 6 \times 4 + 9} = 1$, R.H.S. = $9 - 2 \times 4 = 1$

($\checkmark$)

When $x = 6$

L.H.S. = $\sqrt{6^2 - 6 \times 6 + 9} = 3$, R.H.S. = $9 - 2 \times 6 = -3$

(refused)

$\therefore$ Solution set = $\{4\}$

Find algebraically in $\mathbb{R}$ the S.S of each of the following equations:

(1) $|3x - 2| - 7 = 0$ Sol.

(2) $|x| + 5 = 0$ Sol.

(3) $|3x - 5| = 10$ Sol.

(4) $|x - 2| = 2x + 4$ Sol.

(5) $|x + 3| - 3x + 10 = 0$ Sol.

(6) $x + |x| = 2$ Sol.

$$\begin{aligned}\therefore |x| &= 2 - x \\ \therefore x &= \pm(2 - x)\end{aligned}$$

$$\begin{aligned}x &= 2 - x \\ x + x &= 2 \\ 2x &= 2 \\ x &= 1\end{aligned}$$

$$\therefore S.S = \{1\}$$

(Right)

$$\begin{aligned}x &= -2 + x \\ x - x &= -2 \\ 0 &= -2 \quad (\text{wrong})\end{aligned}$$

(7) $x^2 - 5|x| + 6 = 0$ Sol.

$$\begin{aligned}\therefore |x|^2 - 5|x| + 6 &= 0 \\ \therefore (|x| - 2)(|x| - 3) &= 0\end{aligned}$$

$$\begin{array}{l|l}(|x| - 2) = 0 & (|x| - 3) = 0 \\ |x| = 2 & |x| = 3 \\ x = \pm 2 & x = \pm 3\end{array} \quad \begin{array}{l} \text{(Right)} \\ \text{(Right)} \end{array}$$

$$\therefore S.S = \{2, -2, 3, -3\}$$

(8) $|x + 7| = |2x + 3|$ Sol.

(9) $|x - 1| = |x - 2|$ Sol.

(10) $|x - 1| - 2|2 - x| = 0$ Sol.

(11) $3|x - 5| - |10 - 2x| = 3$ Sol.

Second: Inequality:**Properties of inequality:**

- 1) If $|x| \leq a, a > 0$, then $-a \leq x \leq a$
- 2) If $|x| \geq a, a > 0$, then $x \geq a, x \leq -a$

Find algebraically in R the S.S. of each of the following inequalities:

(1) $|3x + 7| \leq 8$

Sol.

$$\begin{aligned} \therefore -8 &\leq 3x + 7 \leq 8 \\ -8 - 7 &\leq 3x \leq 8 - 7 \\ -15 &\leq 3x \leq 1 \\ \frac{-15}{3} &\leq x \leq \frac{1}{3} \\ -5 &\leq x \leq \frac{1}{3} \end{aligned}$$

$$\therefore S.S = \left[-5, \frac{1}{3}\right]$$

(2) $|3x - 5| \leq 10$

Sol.

(3) $|3x - 2| < 4$

Sol.

$$\begin{aligned} -4 &< 3x - 2 < 4 \\ -4 + 2 &< 3x < 4 + 2 \\ -2 &< 3x < 6 \\ \frac{-2}{3} &\leq x < 2 \end{aligned}$$

$$\therefore S.S = \left] \frac{-2}{3}, 2 \right[$$

(4) $|4 - x| < 7$

Sol.

(5) $|3x + 2| + 5 < 4$

Sol.

$$\therefore |3x + 2| < 4 - 5$$

$$|3x + 2| < -1$$

$$\therefore S.S = \emptyset$$

(6) $|2x - 3| + 3 > 7$

Sol.

(7) $|3x - 7| \geq 2$

Sol.

(8) $\sqrt{x^2 - 6x + 9} \geq 8$

Sol.

(9) $|3 - 2x| > 5$

Sol.

(10) $\frac{1}{|2x - 3|} \geq 2$

Sol.

(11) $\frac{4}{|x - 2|} \geq 2$

Sol.

(12) $|3x - 2| + 2|6x - 4| \geq 20$

Sol.

Find graphically in R the S.S. of each of the following:

(1) $|x - 2| < 3$

Sol.

Put : $f(x) = |x - 2|$

$\therefore$ Point of symmetry = $(2,0)$

Put : $g(x) = 3$

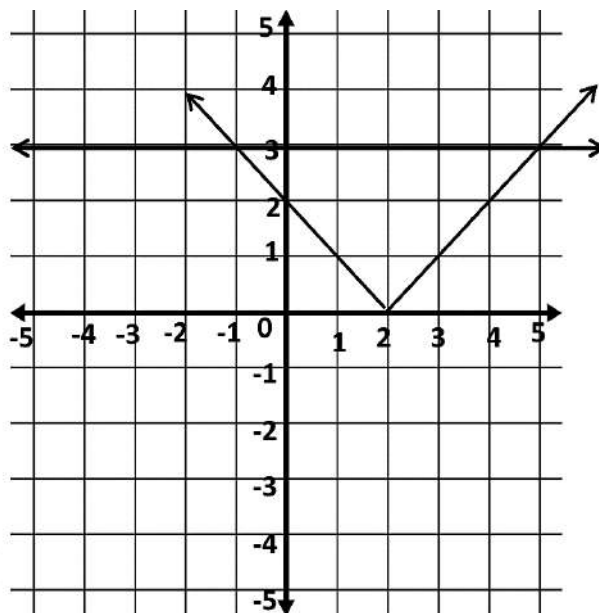
$\therefore$ S.S = $] -1, 5[$

Notice :

1) If $|x - 2| \leq 3 \quad \therefore$ S.S = $[-1, 5]$

2) If $|x - 2| > 3 \quad \therefore$ S.S = $R - [-1, 5]$

3) If $|x - 2| \geq 3 \quad \therefore$ S.S = $R -] -1, 5[$



(2) $|x + 3| > -1$

Sol.

Put : $f(x) = |x + 3|$

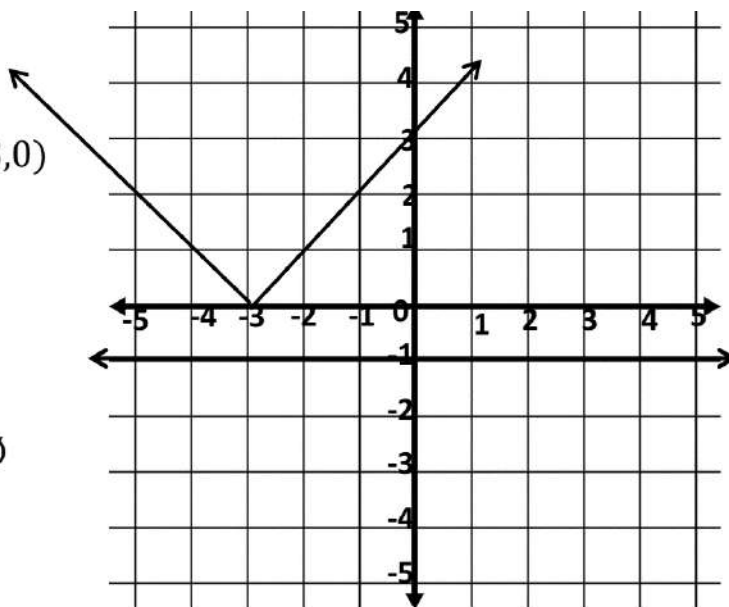
$\therefore$ Point of symmetry = $(-3, 0)$

Put : $g(x) = -1$

$\therefore$ S.S = R

Notice :

If $|x + 3| < -1 \quad \therefore$ S.S = $\emptyset$



Exercises (6)

Choose the correct answer:

1)	The solution set of the equation $ x - 2 = 3$ is	(a) $\{2, 3\}$	(b) $\{-1, 5\}$	(c) $[-1, 5]$	(d) $\{5, -5\}$
2)	The solution set of the equation $ 5x - 1 + 4 = 1$ in $\mathbb{R}$ is	(a) $\{-\frac{4}{5}\}$	(b) $\{-\frac{4}{5}, \frac{6}{5}\}$	(c) $\emptyset$	(d) $\{\frac{6}{5}\}$
3)	The solution set of the equation $ x - 3 = 3 - x $ in $\mathbb{R}$ is	(a) $\{3\}$	(b) $\{3, -3\}$	(c) $\mathbb{R}$	(d) $\emptyset$
4)	The solution set of the equation : $ 2x - 1 = x + 2$ is	(a) $\{3\}$	(b) $\{3, \frac{1}{3}\}$	(c) $\{3, -\frac{1}{3}\}$	(d) $\emptyset$
5)	The solution set of the equation : $\frac{1}{ x-3 } = \frac{1}{2}$ is where $x \neq 3$	(a) $\{5\}$	(b) $\{1\}$	(c) $\{5, 1\}$	(d) $\emptyset$
6)	The solution set of the equation : $\frac{ x }{x} = 1$ in $\mathbb{R}$ is	(a) $\{0\}$	(b) $\mathbb{R}^+$	(c) $\mathbb{R}^-$	(d) $[0, \infty[$
7)	The solution set of the equation $\sqrt{x^2 - 3x} = 8$ is	(a) $\emptyset$	(b) $\{-2, -4\}$	(c) $\{-4\}$	(d) $\{-2\}$
8)	If the solution set of the equation : $ x - a = b$ is $\{5, 9\}$, then $(a, b) =$	(a) $(-2, 7)$	(b) $(7, 2)$	(c) $(2, 7)$	(d) $(-7, 2)$

9)	If $f(x) = \sqrt{x^2} + x $, then the solution set of the equation $f(x) = 2$ in $\mathbb{R}$ is	(a) $\{2, -2\}$	(b) $\{1, -1\}$	(c) $\{2, 0\}$	(d) $\{-1, 0\}$
10)	If $f(x) = x - 2 + 4$, then the solution set of the equation $f(x + 2) = 3$ is	(a) $\{1, 3\}$	(b) $\mathbb{R}$	(c) $\emptyset$	(d) $\{-1, -3\}$
11)	The solution set of the equation $\sqrt{4x^2 - 12x + 9} = 5$ is	(a) $\{4\}$	(b) $\{-1\}$	(c) $\{4, -1\}$	(d) $\mathbb{R}$
12)	The domain of the function $f : f(x) = \frac{1}{ x + 3}$ is	(a) $\mathbb{R}$	(b) $\{3, -3\}$	(c) $\mathbb{R} - \{3, -3\}$	(d) $\mathbb{R} - \{3\}$
13)	If the domain of the function $f : f(x) = \frac{x}{ x + a}$ is $\mathbb{R} - \{2, -2\}$, then $a =$	(a) 2	(b) -2	(c) ± 2	(d) zero
14)	The solution set of the inequality $ x > -1$ is	(a) $[0, \infty[$	(b) $\mathbb{R}$	(c) $\emptyset$	(d) $\mathbb{R} - \{0\}$
15)	The solution set of the inequality : $ 3 - x > 0$ is	(a) $]-3, 3[$	(b) $\mathbb{R} - [-3, 3]$	(c) $\mathbb{R} - \{3\}$	(d) $\mathbb{R}$
16)	The solution set of the inequality $ x + 3 \leq 0$ is	(a) $\emptyset$	(b) $]-\infty, -3]$	(c) $]-3, \infty[$	(d) $\{-3\}$
17)	The solution set of the inequality $ x - 2 \leq -4$ in $\mathbb{R}$ is	(a) $]-2, 6[$	(b) $[-2, 6]$	(c) $\mathbb{R}$	(d) $\emptyset$

18)	The solution set of the inequality : $\sqrt{x^2 - 2x + 1} \geq 4$ is (a) $[-3, 5]$ (b) $\mathbb{R} -]-3, 5[$ (c) $] -3, 5[$ (d) $\mathbb{R} - [-3, 5]$
19)	The solution set of the inequality : $ 2x - 3 + 6 - 4x \leq 21$ is (a) $[-2, 5]$ (b) $\mathbb{R} -]-2, 5[$ (c) $] -2, 5[$ (d) $\{-2, 5\}$
20)	The solution set of the inequality $\sqrt{(x + 2)^2} + 2x + 4 > 6$ is (a) $[-4, 0]$ (b) $\mathbb{R} - [-4, 0]$ (c) $] -4, 0[$ (d) $\mathbb{R} -]-4, 0[$

Unit two

Exponents, Logarithms, and Their Applications

Lesson 1: Rational exponents and exponential equations

Lesson 2: exponential function and its applications

Lesson 3: The inverse function

Lesson 4: Logarithmic function and its graph

Lesson 5: Properties of logarithms

Lesson 1: Rational exponents and exponential equations

Exponential Rules:

If a, b are two real numbers, m, n are two rational numbers, then:

1) $a^{\text{zero}} = 1$

2) $a^m \times a^n = a^{m+n}$

3) $\frac{a^m}{a^n} = a^{m-n}$

4) $(a^m)^n = a^{mn}$

5) $(ab)^n = a^n b^n$

6) $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

7) $a^{-n} = \frac{1}{a^n}$

8) $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n = \frac{b^n}{a^n}$

Remarks:

- $\sqrt[n]{a^n} = |a|$ if n is an even number.
- $\sqrt[n]{a^n} = a$ if n is an odd number.

For example:

- $\sqrt[4]{(-5)^4} = |-5| = 5$
- $\sqrt[4]{x^4} = |x|$
- $\sqrt[3]{(-5)^3} = -5$
- $\sqrt[5]{x^5} = x$

Rational exponents:

1) If $n \in \mathbb{Z}^+ - \{1\}$, $a \in \mathbb{R}$, then $\sqrt[n]{a} = a^{\frac{1}{n}}$

- Take care $\rightarrow$ if n is even number, $a < 0$ then $a^{\frac{1}{n}} = \sqrt[n]{a} \notin \mathbb{R}$

For example:

- $9^{\frac{1}{2}} = \sqrt{9} = 3 \in \mathbb{R}$
- $\left(\frac{-1}{8}\right)^{\frac{1}{3}} = \sqrt[3]{\frac{-1}{8}} = -\frac{1}{2} \in \mathbb{R}$
- $(-16)^{\frac{1}{4}} = \sqrt[4]{-16} \notin \mathbb{R}$

2) If m, n are two integers with no common factor $n > 1$, $\sqrt[n]{a} \in \mathbb{R}$

, Then $a^{\frac{m}{n}} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}$

For example:

$$\bullet 8^{\frac{2}{3}} = (\sqrt[3]{8})^2 = 2^2 = 4$$

$$\bullet \frac{(8)^{-3} \times (18)^2}{(81) \times (16)^{-2}} \bullet (81)^{\frac{-3}{4}} = (\sqrt[4]{81})^{-3} = 3^{-3} = \frac{1}{27}$$

Simplify:

Sol.

$$= \frac{(2^3)^{-3} \times (2 \times 3^2)^2}{(3)^4 \times (2^4)^{-2}} = \frac{2^{-9} \times 2^2 \times 3^4}{3^4 \times 2^{-8}} = 2^{-9+2+8} = 2$$

Simplify: $\frac{9^{4n+1} \times 4^{2-2n}}{3^{9n+1} \times 48^{1-n}}$

Sol.

Prove that: $\frac{125 \times \sqrt[8]{4^3} \times 10^{\frac{-1}{4}}}{\frac{5}{48} \times \sqrt[4]{6^{-3}} \times 15^{\frac{3}{4}}} = 25$

Sol.

Prove that: $\frac{5 \times 3^{2n} - 4 \times 3^{2n-1}}{2 \times 3^{2n+1} - 3^{2n}} = \frac{11}{15}$

Sol.

Exponential Equations:

Rules:

For $m, n \in \mathbb{Z}$ and $a, b \in \mathbb{R} - \{1, 0, -1\}$

(1) If $a^n = 1$, then $n = \text{zero}$

For example: $2^{x+5} = 1 \rightarrow x + 5 = 0 \therefore x = -5$

(2) If $a^m = a^n$, then $m = n$

For example: $2^x = 8 \rightarrow 2^x = 2^3 \therefore x = 3$

(3) If $a^n = b^n$, then :

□ $a = b$ when n is odd

□ $a = \pm b$ when n is even

□ $n = 0$ when $a \neq b$

For example:

$$x^5 = 32 \rightarrow x^5 = 2^5 \rightarrow \therefore x = 2$$

$$x^2 = 16 \rightarrow x^2 = 4^2 \text{ Or } x^2 = (-4)^2 \rightarrow \therefore x = \pm 4$$

$$2^{x+1} = 3^{x+1} \rightarrow x + 1 = 0 \rightarrow x = -1$$

Find the value of x that satisfies each of the following equations:

(1) $2^{x-3} = 16$

(2) $3^{x+1} = \frac{1}{27}$

(3) $5^{x^2-9} = 1$

Sol.

(4) $\left(\frac{2}{3}\right)^{x+3} = \left(3\frac{3}{8}\right)^{-3}$

(5) $3^{x-4} = 8^{x-4}$

(6) $2^{x+4} = x^{x+4}$

Sol.

(7) $(3\sqrt{3})^{x+2} = 81$

(8) $(2\sqrt{2})^{x-3} = 8^{x-2}$

(9) $\frac{(3^2+5^2)^x}{68^x} = \frac{1}{32}$

Sol.

$$(10) 5^{x-2} = 2^{3x-6}$$

$$(11) \frac{3^{3x} + 3^{2x} + 3^x}{3^{2x} + 3^{x+1}} = \frac{1}{27}$$

$$(12) 2^x \times \sqrt[3]{4} = (\sqrt[3]{16})^{-1}$$

Sol.

$$(13) 4^{x^2-1} = 8^{-x}$$

$$(14) 2^{x+1} + 2^{x-1} = 5$$

$$(15) 5^x + \frac{125}{5^x} = 30$$

Sol.

Exercises (1)

Choose the correct answer:

1)	$\sqrt[5]{a^3} \times \sqrt{a^3} = \dots\dots\dots$ (a) $\sqrt[7]{a^3}$ (b) $\sqrt[7]{a^6}$ (c) $\sqrt[7]{a^{14}}$ (d) $a^2 \sqrt[10]{a}$
2)	 If $2^{x+1} = 8$, then $x = \dots\dots\dots$ (a) 1 (b) 2 (c) 3 (d) 4
3)	If $3^{x+5} = \frac{1}{27}$, then $x = \dots\dots\dots$ (a) -3 (b) 8 (c) -8 (d) 3
4)	 If $5^{x-1} = 4^{x-1}$, then $x = \dots\dots\dots$ (a) 5 (b) 1 (c) -1 (d) zero
5)	The solution set of the equation : $5^{x^2-4} = 7^{x^2-4}$ is $\dots\dots\dots$ (a) {2} (b) {-2} (c) {2, -2} (d) {zero}
6)	If $7^{x+1} = 3^{2x+2}$, then $5^{x+1} = \dots\dots\dots$ (a) zero (b) 1 (c) 2 (d) 5
7)	 If $\left(\frac{1}{2}\right)^{a^2-a-2} = 1$ where $a > \text{zero}$, then $a = \dots\dots\dots$ (a) 1 (b) -3 (c) 2 (d) 3
8)	If $3^x = 2$, $2^y = 9$, then $xy = \dots\dots\dots$ (a) 2 (b) 3 (c) 8 (d) 18

9)	📖 If $5^x = 2$, then $(25)^x = \dots\dots\dots$ (a) 10 (b) 625 (c) 4 (d) 2
10)	📖 If $x^{\frac{3}{2}} = 64$, then $x = \dots\dots\dots$ (a) 512 (b) 16 (c) 4 (d) 2
11)	📖 If $4x^5 = 128$, then $x = \dots\dots\dots$ (a) 4 (b) ± 2 (c) 2 (d) -2
12)	📖 $\sqrt[4]{(16)^{-3}} = \dots\dots\dots$ (a) 8 (b) -8 (c) $\frac{1}{8}$ (d) $-\frac{1}{8}$
13)	📖 $\sqrt[4]{16x^4y^8} = \dots\dots\dots$ (a) $2xy^2$ (b) $2 x y^2$ (c) $2x y ^2$ (d) $2x y^2$
14)	If $x^{\frac{5}{3}} = 2$, $y^{\frac{4}{3}} = 32$, then $x + y = \dots\dots\dots$ (a) 16 (b) zero. (c) 16 , -16 (d) zero , 16
15)	📖 The solution set of the equation : $3^{x+1} + 3^x = 12$ in $\mathbb{R}$ is $\dots\dots\dots$ (a) $\{0\}$ (b) $\{3\}$ (c) $\{1\}$ (d) $\{1, 0\}$
16)	The solution set of the equation : $\sqrt[3]{x^2} - 3\sqrt[3]{x} + 2 = 0$ is $\dots\dots\dots$ (a) $\{1, 8\}$ (b) $\{9, 3\}$ (c) $\{8\}$ (d) $\{1\}$
17)	📖 The set of the real roots of the equation : $(x-2)^4 = 16$ equals $\dots\dots\dots$ (a) $\{0\}$ (b) $\{4\}$ (c) $\{8\}$ (d) $\{0, 4\}$

18)	The solution set of the equation : $(x - 3)^{\frac{5}{3}} = 32$ in $\mathbb{R}$ is	
(a) $\{2\}$	(b) $\{11\}$	(c) $\{11, -5\}$ (d) $\{-11, 11\}$
19)	 Which of the following is not equal to $\sqrt[5]{x^4}$?	
(a) $(\sqrt[5]{x})^4$	(b) $\sqrt[4]{x^5}$	(c) $x^{\frac{4}{5}}$ (d) $(x^{\frac{1}{5}})^4$
20)	 The number $(2^{24} + 2^{23} + 2^{22})$ is divisible by	
(a) 3	(b) 5	(c) 7 (d) 9
21)	If $3^{x-2} = \sqrt[4]{27}$, then $x =$	
(a) $\frac{11}{4}$	(b) $\frac{4}{3}$	(c) $\frac{3}{4}$ (d) 6
22)	The number of real roots of the equation : $x^4 = -16$ is	
(a) zero	(b) 1	(c) 2 (d) 4

Lesson 2: exponential function and its applications

The function $f(x) = a^x$ is called an exponential function, such that:

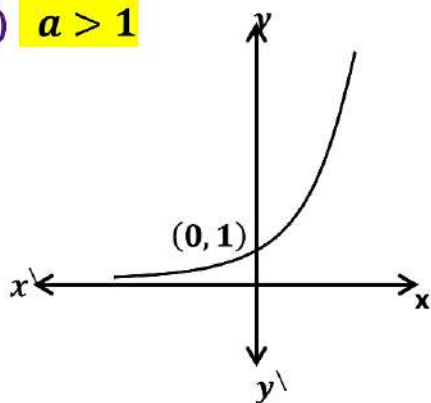
$$\diamond a \in \mathbb{R}^+ - \{1\}$$

$$\diamond x \in \mathbb{R}$$

Graphical representation of the exponential function:

The general form is : $f(x) = a^x$

1) $a > 1$



1) Domain = $\mathbb{R}$

2) Range = $\mathbb{R}^+ =]0, \infty[$

3) The fn. is increasing on its domain

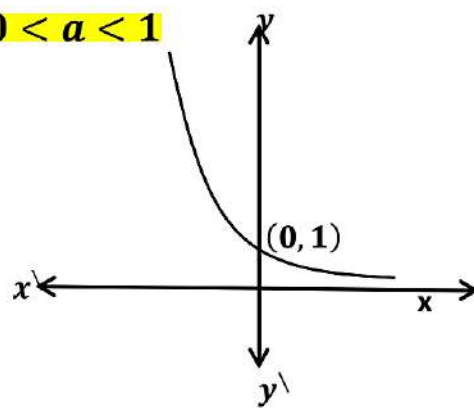
4) The curve of the fn. passes through
The point (0,1)

5) The curve of the fn. approaches
The **-ve direction** of x-axis

6) Is called : **exponential growth fn.**

7) The fn. is one-to-one

2) $0 < a < 1$



1) Domain = $\mathbb{R}$

2) Range = $\mathbb{R}^+ =]0, \infty[$

3) The fn. is decreasing on its domain

4) The curve of the fn. passes through
The point (0,1)

5) The curve of the fn. approaches
The **+ve direction** of x-axis

6) Is called : **exponential decay fn.**

7) The fn. is one-to-one

Note : The curve of $y = \left(\frac{1}{a}\right)^x$ is the image of the curve of $y = a^x$ by reflection in y -axis
(equation of y -axis is $X = 0$)

For example (1):

Graph the function $f(x) = 2^x$ where $x \in [-3, 3]$, then from the graph find :

1) $f(1.5)$

2) If $f(x) = 6$, Find x

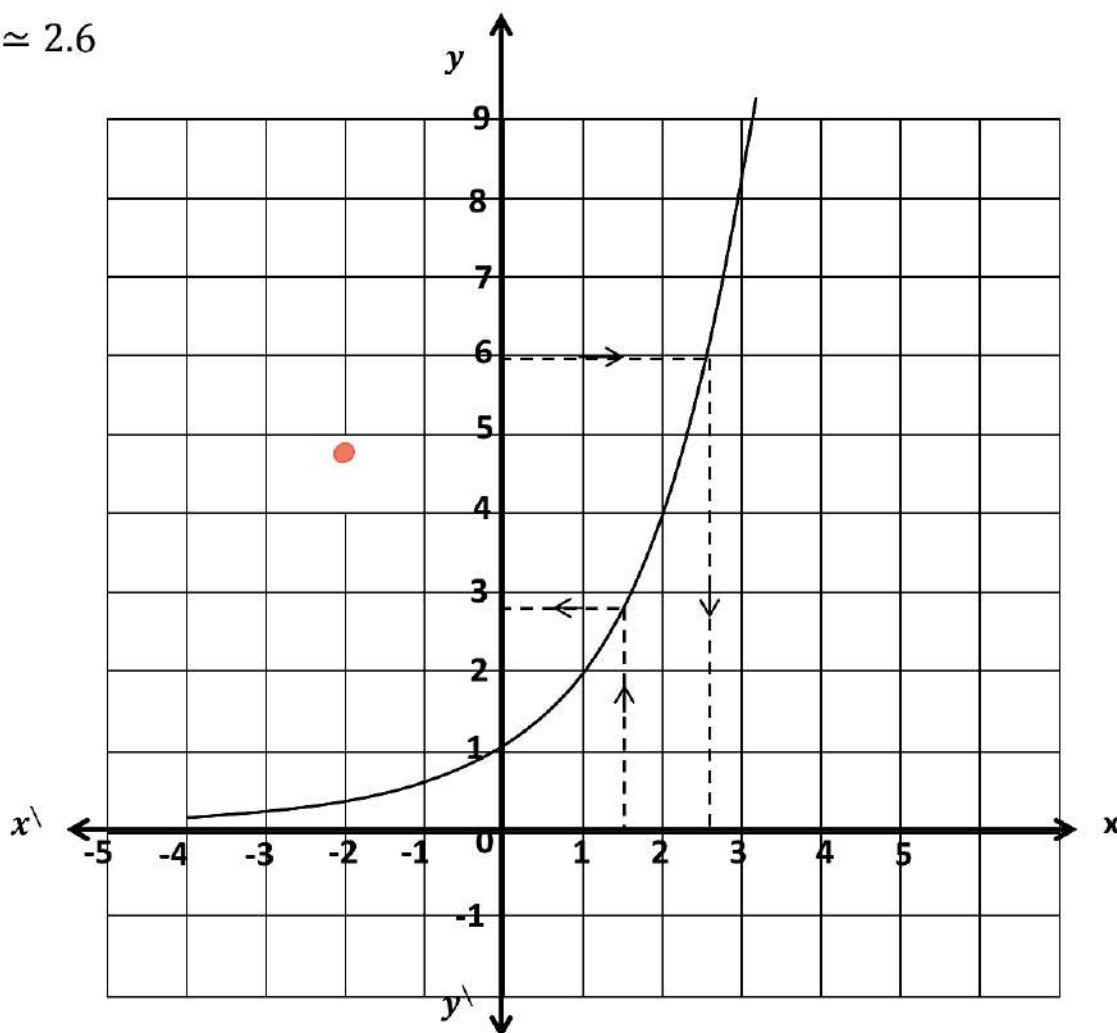
Sol.

x	-3	-2	-1	0	1	2	3
F(x)	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

1) $f(1.5) \simeq 2.8$

2) $\because 2^x = 6$

$\therefore x \simeq 2.6$



For example (2):

Graph the function $f(x) = \left(\frac{1}{2}\right)^x$, where $x \in [-3, 3]$

Then from the graph find :

- 1) $f(0.5)$
- 2) The value of x when $\left(\frac{1}{2}\right)^x = 5.3$

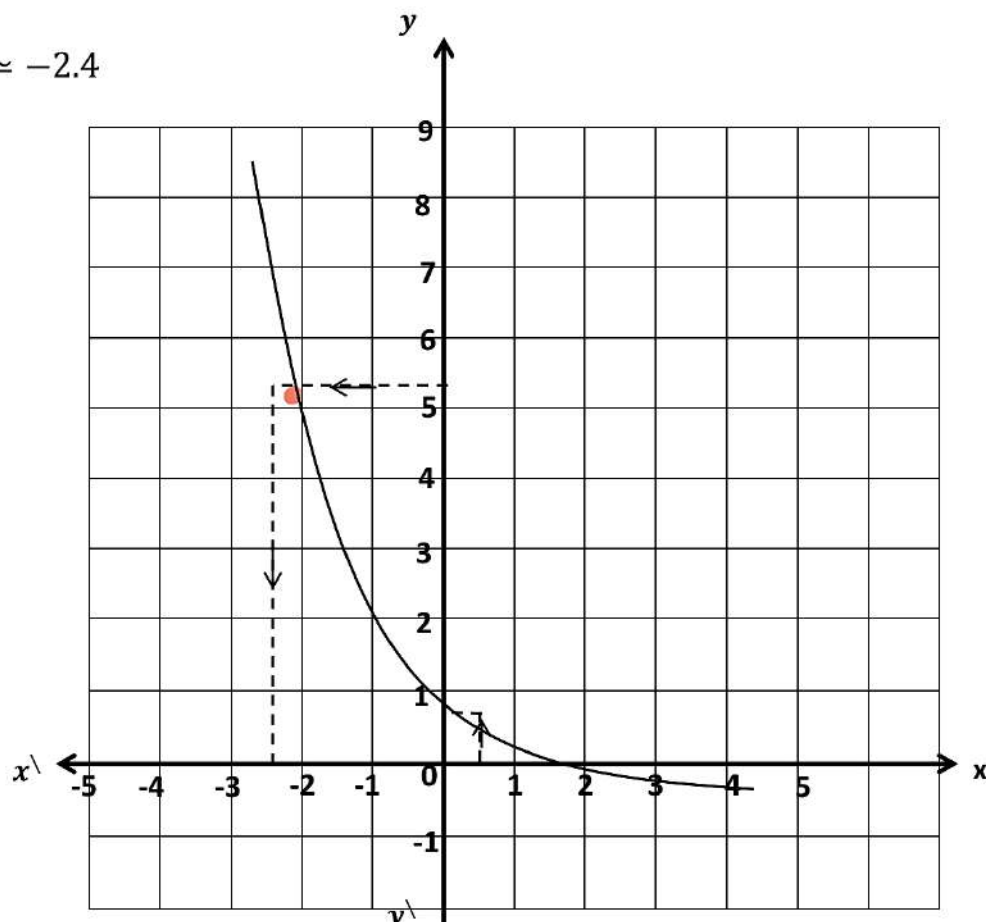
Sol.

x	-3	-2	-1	0	1	2	3
F(x)	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

1) $f(0.5) \simeq 0.7$

2) $\because \left(\frac{1}{2}\right)^x = 5.3$

$\therefore x \simeq -2.4$



(3) If $f(x) = 3^x$ then complete the following:

(a) $f(2) = \dots\dots\dots$

(b) $f(x + 2) = \dots\dots \times f(x)$

(c) $f(x) \times f(-x) = \dots\dots\dots$

(4) If $f(x) = 7^x$, $f(x) + f(x - 1) = 56$. Find the value of x .

[5] If $f(3) = 3^x$

(a) Prove that: $f(x + 2) \times f(x - 2) = f(2x)$

(b) If $f(x + 1) - f(x - 1) = 72$. Find x

[6] If $f_1(x) = 8^x$, $f_2(x) = 4^x$

(a) prove that: $\frac{f_1(2x+1)+f_2(3x+2)}{f_1(2x-1)+f_2(3x-2)} = 128$

(b) Solve the equation: $f_1(2x) + f_2(3x-1) = 80$

[7] If $f(x) = 2^x$,

(a) find x which satisfies the equation: $f(x) + f(5 - x) = 12$

(b) $\frac{f(x+1)}{f(x-1)} + \frac{f(x-1)}{f(x+1)} = \frac{17}{4}$

Exercises (2)

Choose the correct answer:

1)	If $f : f(x) = a^x$ is an exponential function, then $a \in \dots\dots\dots$ (a) $\mathbb{R}$ (b) $\mathbb{R}^+$ (c) $\mathbb{R}^-$ (d) $\mathbb{R}^+ - \{1\}$
2)	If $f : f(x) = 3^{x+2}$, then $f(-2) = \dots\dots\dots$ (a) 3 (b) zero (c) -1 (d) 1
3)	If $f(x) = 4^{x-1}$, then $f(x+1) = \dots\dots\dots$ (a) 4^x (b) 4^{x+1} (c) 4^{x+2} (d) 2^x
4)	If $f(x) = 2^x$, then $f(-x) = \dots\dots\dots$ (a) -2^x (b) $\left(\frac{1}{2}\right)^x$ (c) 2^{x+1} (d) $\left(\frac{1}{2}\right)^{-x}$
5)	If $f(x) = (5)^{-x}$, then $\frac{f(x-1)}{f(x+1)} = \dots\dots\dots$ (a) 5 (b) $\frac{1}{5}$ (c) 25 (d) $\frac{1}{25}$
6)	If $f(x) = a^x$, then $f(x+1) \times f(x-1) = f(\dots\dots\dots)$ (a) $2x+1$ (b) a^{2x} (c) $2x$ (d) 2
7)	If $f(x) = 3^{x-2}$, then the solution set of the equation : $f(x-1) = 81$ is $\dots\dots\dots$ (a) $\{7\}$ (b) $\{5\}$ (c) $\{4\}$ (d) $\{3\}$
8)	If $f(x) = 2^x$, then the solution set of the equation $f(2x) - f(x+1) = \text{zero}$ is $\dots\dots\dots$ (a) $\{0\}$ (b) $\{0, 1\}$ (c) $\{1\}$ (d) $\{-1\}$

9)	If $f(x) = 3^x$, then the value of x which satisfy the equation : $f(x+1) - f(x-1) = 24$ is (a) 2 (b) 3 (c) 8 (d) zero.
10)	 The exponential function of base a is increasing if (a) $a > 0$ (b) $a > 1$ (c) $0 < a < 1$ (d) $a = 1$
11)	 The exponential function of base a is decreasing if (a) $a > 0$ (b) $a < 0$ (c) $0 < a < 1$ (d) $-1 < a < 0$
12)	 The equation of the symmetry axis of the two functions f, g where $f(x) = 3^x$ and $g(x) = \left(\frac{1}{3}\right)^x$ is (a) $y = 0$ (b) $x = 0$ (c) $y = x$ (d) $y = -x$
13)	The curve of the function $f : f(x) = 5^x$ intersects the y-axis at the point (a) (1 , 0) (b) (0 , 1) (c) (5 , 1) (d) (1 , 5)
14)	The curve of the function $f : f(x) = 2^{x+2}$ intersects the y-axis at the point (a) (0 , 1) (b) (0 , 2) (c) (0 , 4) (d) (0 , 8)
15)	 The exponential function f where $f(x) = a^x, a > 1$, its curve approaches (a) the x-axis (positive direction) (b) the x-axis (negative direction) (c) the y-axis (positive direction) (d) the y-axis (negative direction)
16)	 In the exponential function $f : f(x) = a^x, a > 1$, then $f(x) > 1$ when $x \in$ (a) $\mathbb{R}$ (b) $\mathbb{R}^+$ (c) $\mathbb{R}^-$ (d) $\mathbb{Z}$
17)	 In the exponential function g where $g(x) = a^x, 0 < a < 1$, then $0 < a^x < 1$ when $x \in$ (a) $]0, \infty[$ (b) $]-\infty, 0]$ (c) $]1, \infty[$ (d) $]-\infty, 1]$

Lesson 3: The inverse function

If the function f is a one-to-one function from x to y , then f^{-1} is an inverse function of f from y to x if:

For $(x, y) \in f$



For $(y, x) \in f^{-1}$



For example, if the function f is as follows:

$f = \{(1, 2), (2, 4), (3, 6), (4, 8)\}$. Find the inverse function of f and represent both in one figure.

The function f is one-to-one, so it has an inverse

$$\therefore f = \{(1, 2), (2, 4), (3, 6), (4, 8)\}$$

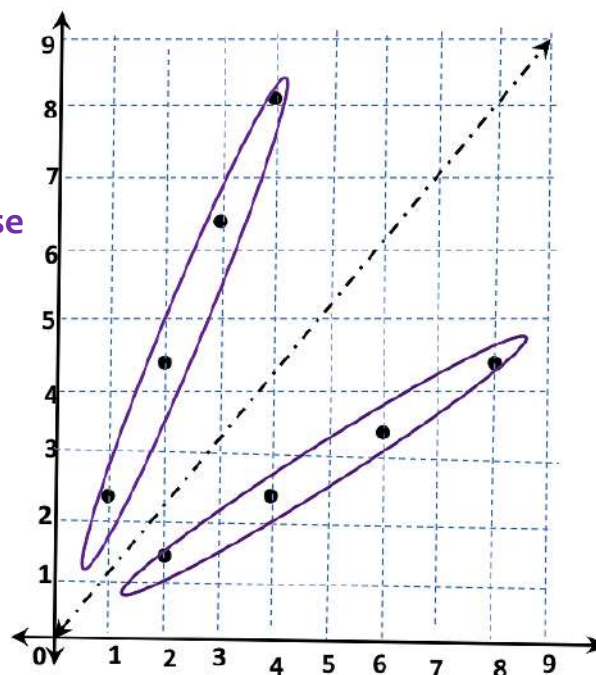
$$\therefore f^{-1} = \{(2, 1), (4, 2), (6, 3), (8, 4)\}$$

We notice that the function f and the inverse

f^{-1} is symmetric about the straight-line $y = x$

Thus $f^{-1}(x)$ is the image of $f(x)$ by reflection in

The straight-line $y = x$.



Properties of the inverse function

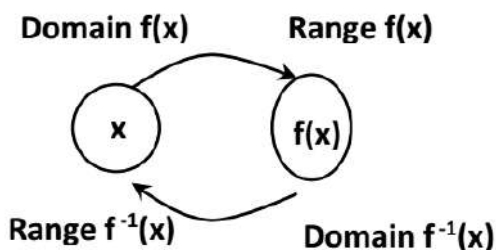
(1) we said that $f(x)$, $g(x)$ each one is inverse

function to the other if: $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$

(2) The domain of $f(x)$ = the range of inverse

function $f^{-1}(x)$ and the range of $f(x)$ = the domain

of the inverse function.



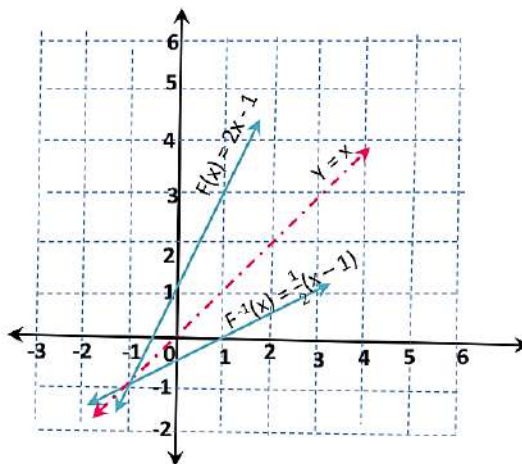
Examples

[1] Find the inverse function of the function f such that $f(x) = 2x + 1$ and represent $f(x)$ and its inverse graphically in one figure.

Sol.

$$\begin{aligned} \therefore y &= 2x + 1 && \text{(exchange variables)} \\ \therefore x &= 2y + 1 && \therefore 2y = x - 1 \\ \therefore y &= \frac{1}{2}(x - 1) && \therefore f^{-1}(x) = \frac{1}{2}(x - 1) \end{aligned}$$

x	1	0	-1
$f(x)$	3	1	-1
$f^{-1}(x)$	0	$\frac{1}{2}$	-1



Notice that: the curves of the function f and f^{-1} are symmetric about the straight line $y = x$

[2] If $f(x) = \frac{1}{x-3} + 2$, find: (a) the domain and the range of the function f
(b) $f^{-1}(x)$ and find its domain and its range

Sol.

(a) the domain of $f(x) = \mathbb{R} - \{3\}$, and the range of $f(x) = \mathbb{R} - \{2\}$

(b) $\therefore y = \frac{1}{x-3} + 2$ by exchange the variables x, y

$$\therefore x = \frac{1}{y-3} + 2 \quad \therefore \frac{1}{y-3} = x - 2 \quad \therefore y - 3 = \frac{1}{x-2}$$

$$\therefore y = \frac{1}{x-2} + 3 \quad \therefore f^{-1} = \frac{1}{x-2} + 3$$

$\therefore$ the domain of $f(x) = \mathbb{R} - \{2\}$, and the range of $f(x) = \mathbb{R} - \{3\}$

[3] Prove that f, g where: $f(x) = 4x + 9, g(x) = \frac{x-9}{4}$ inverse to each other

Sol.

$$\therefore (f \circ g)(x) = f[g(x)] = f\left(\frac{x-9}{4}\right)$$

$$\therefore (f \circ g)(x) = 4\left(\frac{x-9}{4}\right) + 9 = (x-9) + 9 = x \quad \dots\dots\dots (1)$$

$$\therefore (g \circ f)(x) = g[f(x)] = g[4x + 9]$$

$$\therefore (g \circ f)(x) = \frac{4x+9-9}{4} = \frac{4x}{4} = x \quad \dots\dots\dots (2)$$

$\therefore$ From (1) and (2) f, g are inverse to each other

[4] Find the inverse function of the function f where $f(x) = (x - 1)^2 + 2$, $x \leq 1$, then find the domain of $f^{-1}(x)$

[5] If $f(x) = 3 + \sqrt{x - 1}$ find:

- (a) The domain and the range of the function f
- (b) $f^{-1}(x)$ and find its domain and its range

Exercises (3)

Choose the correct answer:

1)	If f is an odd function and g is a function where the curve of g is the image of the curve of f by reflection in the straight line $y = x$, then (a) $g(x) = f(x)$ (b) $g(x) = \frac{1}{f(x)}$ (c) $g(x) = f^{-1}(x)$ (d) $g(x) = (f \circ f)(x)$	
2)	Which of the following does not have an inverse function ? (a) $y = \sqrt{x}$ (b) $y = 3x$ (c) $y = x^3$ (d) $y = x^2$	
3)	The function f, f^{-1} are image of each other by reflection in the straight line (a) $y = 0$ (b) $x = 0$ (c) $y = -x$ (d) $y = x$	
4)	If $(a, b) \in$ the curve of the function f, then $\in$ the curve of the function f^{-1} (a) (a, b) (b) $(a, -b)$ (c) (b, a) (d) $(b, -a)$	
5)	If $f(7) = 3$, then $f^{-1}(3) =$ (a) 3 (b) 4 (c) 7 (d) 10	
6)	If the function $f = \{(1, 4), (2, -3), (3, 1), (4, 0)\}$, then $f^{-1}(1) + f(2) =$ (a) -1 (b) zero (c) 1 (d) 3	
7)	If the function f^{-1} where $f^{-1} = \{(2, 3), (5, b)\}$ is the inverse function of the function f where $f = \{(4, 5), (a, 2)\}$, then $a - b =$ (a) zero (b) 1 (c) -1 (d) 2	
8)	If f is a function where $f(x) = 7x$, then $f^{-1}(x) =$ (a) $7x$ (b) $\frac{x}{7}$ (c) $\frac{7}{x}$ (d) $7 - x$	

9)

If $y = \sqrt[3]{x}$, then its inverse function is $y = \dots\dots\dots$

(a) $\frac{1}{3}x^3$

(b) x^3

(c) $x^3 - 1$

(d) $3x^3$

10)

If $f(x) = x^3 + 7$, then $f^{-1}(-1) = \dots\dots\dots$

(a) 1

(b) 2

(c) -2

(d) 8

11)

If $f^{-1}(x) = 2x + 1$, then $f(x) = \dots\dots\dots$

(a) $-2x - 1$

(b) $\frac{1}{2}x + 1$

(c) $\frac{1}{2}x - \frac{1}{2}$

(d) $\frac{1}{2x} + 1$

12)



The opposite figure

represents the function $f : X \longrightarrow Y$,

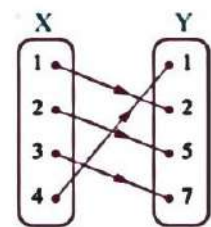
then $f^{-1}(2) = \dots\dots\dots$

(a) 1

(b) 5

(c) 4

(d) 7



Lesson 4: Logarithmic function and its graph

If $a \in \mathbb{R}^+ - \{1\}$ then the function $f: \mathbb{R}^+ \longrightarrow \mathbb{R}$ where $f(x) = \log_a x$ is the inverse function of the exponential function $y = a^x$ and $f(x)$ is called the logarithmic function

- * Domain of the logarithmic function = $\mathbb{R}^+$
- * Range of logarithmic function = $\mathbb{R}$
- * The form $y = \log_a x$ is equivalent to $a^y = x$

For example

$$2^5 = 32 \longrightarrow \log_2 32 = 5$$

$$10^{-3} = 0.001 \longrightarrow \log_{10} 0.001 = -3$$

$$5^3 = 125 \longrightarrow \log_5 125 = 3$$

$$x^y = z \longrightarrow \log_x z = y$$

Note that: If the base of the logarithm is 10, it is named the common logarithm, and written without a base, such as $\log_{10} 7$ is written $\text{Log } 7$

Examples

[1] Find the value of:

(a) $\log_3 \sqrt[4]{27}$

(b) $\log_{\frac{1}{2}} 128$

(c) $\text{Log } 0.00001$

[2] Find in $\mathbb{R}$ the solution set of each of the following equations:

(a) $\log_3 (2x - 5) = 1$

(b) $\log_x (x + 2) = 2$

(c) $\log_{81} x = \frac{3}{4}$

Graphical representation of the logarithmic function

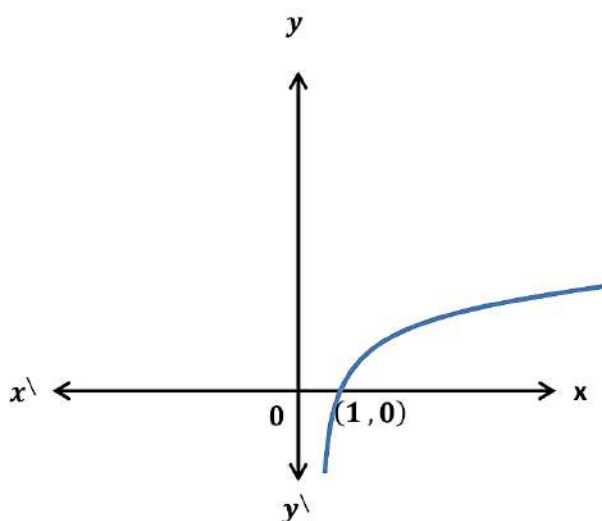
[3] Represent the following functions graphically:

(a) $\log_2 x$

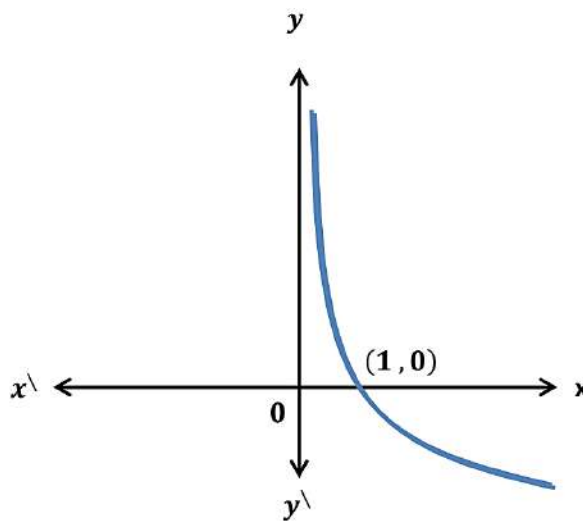
(b) $\log_{\frac{1}{2}} x$

Sol.

x	1	2	4
F(x)	0	1	2



x	1	2	4
F(x)	0	-1	-2



• Properties of the logarithmic function : $f(x) = \log_a x$:

- 1) The domain = R^+
- 2) The range = R
- 3) If $a > 1$, then the function is increasing,
If $0 < a < 1$, then the function is decreasing.
- 4) The curve passes through the point $(1, 0)$
- 5) The logarithmic function is one – to – one.

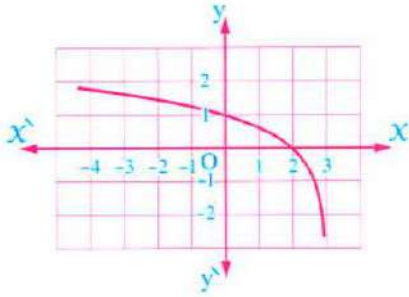
Note that: The logarithmic function is equivalent form of the inverse function for the exponential function

Exercises (4)

Choose the correct answer:

1)	The form $\log_a x = y$ is equivalent to	(a) $\log_a y = x$	(b) $a^y = x$	(c) $a^x = y$	(d) $y = a x$
2)	$\log_{\frac{5}{2}} \frac{16}{625} = \dots\dots\dots$	(a) -2	(b) -4	(c) 3	(d) 5
3)	$\log_3 \log_2 8 = \dots\dots\dots$	(a) -2	(b) -1	(c) 1	(d) 4
4)	If $\log 0.01 = 3x + 1$, then $x = \dots\dots\dots$	(a) -3	(b) -1	(c) 2	(d) 7
5)	If $\log_3 x = 2$, then $x = \dots\dots\dots$	(a) 3	(b) 5	(c) 8	(d) 9
6)	 If $\log (x + 11) = 2$, then $x = \dots\dots\dots$	(a) -9	(b) 22	(c) 89	(d) 91
7)	The S.S. of the equation $\log_x 81 = 4$ in $\mathbb{R}$ is	(a) $\{-3\}$	(b) $\{3\}$	(c) $\{3, -3\}$	(d) $\{9\}$
8)	If $\log_x 25 = 2$, then $x^3 + x^2 - x = \dots\dots\dots$	(a) 95	(b) 105	(c) 145	(d) 155
9)	If $\log_3 (2x + 3) = 2$, then $x = \dots\dots\dots$	(a) 3	(b) 2	(c) 9	(d) 4

10)	The solution set of the equation $\log (X - 1) = \text{zero}$ is	(a) $\left\{ \frac{1}{10} \right\}$	(b) $\{1\}$	(c) $\{2\}$	(d) $\{-1\}$
11)	Solution set of the equations $\log_x (X + 6) = 2$, in $\mathbb{R}$ is	(a) $\{3, -2\}$	(b) $\{3\}$	(c) $\{3, 1\}$	(d) $\{6, 1\}$
12)	If $\log_2 \log_5 \log_2 X = \text{zero}$, then $X = \dots\dots\dots$	(a) 4	(b) 8	(c) 16	(d) 32
13)	If $f(X) = 3^X$, then $f^{-1}(X) = \dots\dots\dots$	(a) $\log_X 3$	(b) $\log_3 X$	(c) X^{-3}	(d) 3^X
14)	The value of $\log_6 33 \approx \dots\dots\dots$ (by using calculator)	(a) 1.95	(b) 0.0512	(c) 2.297	(d) 0.74
15)	The curve of the function $f : f(X) = \log_2 (3 - X)$ intersects the X -axis at the point	(a) (1 , 0)	(b) (2 , 0)	(c) (0 , 1)	(d) (3 , 0)
16)	The range of the function $f : f(X) = \log_2 X$ is	(a) $\mathbb{R}^+$	(b) $\mathbb{R}^-$	(c) $\mathbb{R}$	(d) $\mathbb{R}^*$
17)	The function $f : f(X) = \log_a X$ is decreasing for every $a \in \dots\dots\dots$	(a) $]0, \infty[$	(b) $] - \infty, 0[$	(c) $]0, 1[$	(d) $]1, \infty[$
18)	 If the curve of the function $y = \log_4 (1 - aX)$ passes through the point $\left(\frac{1}{4}, -\frac{1}{2}\right)$, then $a = \dots\dots\dots$	(a) 2	(b) 3	(c) 4	(d) 8

19)	 If the curve of the function f where $f(x) = \log_a x$ passes through the point $(8, 3)$, then $f(4) = \dots\dots\dots$ (a) 1 (b) 2 (c) $\frac{1}{4}$ (d) -2
20)	 The domain of the function f where $f(x) = \log_{(1-x)} 3$ is $\dots\dots\dots$ (a) $]-\infty, 0[\cup]0, 1[$ (b) $]-\infty, 1[$ (c) $]1, \infty[$ (d) $]-1, 1[$
21)	 The domain of the function $f : f(x) = \log_{(1-x)} x$ is $\dots\dots\dots$ (a) $x > 0$ (b) $x < 1$ (c) $0 < x < 1$ (d) $0 \leq x \leq 1$
22)	 The opposite figure represents the function $\dots\dots\dots$ (a) $y = 3^{x-1}$ (b) $y = 3^{x+1}$ (c) $y = \log_3 (2 - x)$ (d) $y = \log_3 (3 - x)$ 

Lesson 5: Properties of logarithms

If $x, y \in \mathbb{R}^+$, $a \in \mathbb{R}^+ - \{1\}$, $m \in \mathbb{R}$, then :

$$(1) \log_a xy = \log_a x + \log_a y$$

$$(3) \log_a x^m = m \log_a x$$

$$(5) \log_a 1 = \text{zero}$$

$$(7) \log_a b = \frac{1}{\log_b a}$$

$$(2) \log_a \frac{x}{y} = \log_a x - \log_a y$$

$$(4) \log_a a = 1$$

$$(6) \log_a b = \frac{\log b}{\log a} = \log_{a^n} b^n$$

Note that:

$$(1) \log_a (x \pm y) \neq \log_a x \pm \log_a y ,$$

$$(2) \log_a xy \neq \log_a x \times \log_a y$$

$$(3) \log_a \frac{xy}{zm} = \log_a x + \log_a y - \log_a z - \log_a m$$

Examples

Without using a calculator, find the value of:

$$(1) \log_3 15 + \log_3 6 - \log_3 10$$

$$(3) \log_4 \frac{2}{5} + \log_4 20 - \log_4 8$$

$$(2) \log_2 15 + \log_2 14 - \log_2 105$$

$$(4) \log_5 125 - \log_5 75 + \log_5 \frac{3}{5}$$

$$(5) \log \frac{6}{5} + \log \frac{225}{4} - \log 243 + \log 5 - \log \frac{5}{36}$$

$$(6) \log_5 100 - 3 \log_5 2 - \log_5 18 + \log_5 36$$

$$(7) 3 \log 5 + 2 \log 14 - \log 49 + \log 0.2$$

$$(8) 3 \log_2 14 - 4 \log_2 5 + 2 \log_2 \frac{25}{7} - \log_2 7$$

$$(9) \log\left(1 + \frac{1}{5}\right) + 2 \log \frac{15}{2} - \log \frac{5}{36} - \log 243 + \log 5$$

$$(10) \frac{\log_2 243 - \log_3 32}{\log_2 27 - \log_3 8}$$

Find the S.S. of the following:

$$(1) \log x - \log 15 = \log 0.2$$

$$(2) \log(x + 5) + \log(x + 8) = 1$$

$$(3) \log_{\frac{1}{5}} \log_5 \sqrt{5x} = 0$$

$$(4) \sqrt{x^{\log \sqrt{x}}} = 10$$

Exercises (5)

Choose the correct answer:

1)	$\log_2 5 \times \log_5 2 = \dots\dots\dots$ (a) 1 (b) 10 (c) $\log_2 10$ (d) $\log_5 10$
2)	$1 + \log 2 = \dots\dots\dots$ (a) $\log 5$ (b) $\log 2$ (c) $\log 20$ (d) $-\log 5$
3)	The value of the expression : $2 \log 25 + \log \frac{8}{15} + 2 \log 3 - \log 30 = \dots\dots\dots$ (a) 6 (b) 2 (c) 3 (d) -1

4)	If $\log X - \log 2 = \log 4$, then $X = \dots\dots\dots$ (a) 4 (b) 6 (c) 8 (d) 16
5)	If $\log X + \log 5 = 2$, then $X = \dots\dots\dots$ (a) 3 (b) 8 (c) 17 (d) 20
6)	$2 \log_5 3 + 3 \log_5 2 = \dots\dots\dots$ (a) $\log_5 6$ (b) $6 \log_5 6$ (c) $\log_5 72$ (d) $\log_5 36$
7)	$\log_b a \times \log_c b \times \log_d c \times \log_a d = \dots\dots\dots$ (a) zero (b) 1 (c) abcd (d) ad
8)	$\sqrt{3^{\log 4} \times 3^{\log 25} + 7^{\log_7 16}} = \dots\dots\dots$ (a) 9 (b) 25 (c) 5 (d) 16
9)	$\frac{1}{\log_2 30} + \frac{1}{\log_3 30} + \frac{1}{\log_5 30} = \dots\dots\dots$ (a) 1 (b) 2 (c) 5 (d) 30
10)	$2 \log_a X + \log_a y - \log_a (Xy) = \dots\dots\dots$ (a) $\log X$ (b) $\log_a X$ (c) $\log_a Xy$ (d) $\log_a X^2$
11)	If $X - 2 = \log_2 3$, then $X = \dots\dots\dots$ (a) $\log_2 6$ (b) $\log_2 9$ (c) $\log_2 12$ (d) $\log_2 18$
12)	 If $3^X = 5$, then $X = \dots\dots\dots$ (a) 3 (b) $\log_3 5$ (c) $\log_5 3$ (d) $\frac{5}{3}$

13)	The solution set of the equation : $\log_2 (2^{x-4}) = 5 - x$ is (a) {4} (b) {4.5} (c) {5} (d) {5.4}
14)	The solution set of the equation : $2 \log 2 - \log x = \log (x+3) - \log 7$ is (a) {7} (b) {4} (c) {7 , 4} (d) $\emptyset$
15)	The solution set of the equation : $\log x^2 - (\log x)^2 = 0$ is (a) {1} (b) {1 , 10} (c) {1 , 100} (d) {100}
16)	If $\log_3 5 = a$, then $\log_{15} 5 =$ (a) a^2 (b) $3 a$ (c) $\frac{1}{a+3}$ (d) $\frac{a}{a+1}$
17)	If $\log 3 = x$, $\log 4 = y$, then $\log 12 =$ (a) $x + y$ (b) $x y$ (c) $x - y$ (d) $\log x + \log y$
18)	The solution set of the equation : $\log_b (x+5) = \log_b x + \log_b 5$ in $\mathbb{R}$ is (a) {5} (b) {4} (c) $\left\{ \frac{4}{5} \right\}$ (d) $\left\{ \frac{5}{4} \right\}$

Unit three

Limits and continuity Differential calculus

- 1 Introduction to limits of functions
- 2 Finding the limit of a function algebraically
- 3 Theorem (4) " the law"
- 4 Limit of the function at infinity
- 5 Limits of trigonometric functions
- 6 Existence of the limit of a piecewise function
- 7 continuity

Lesson (1): Introduction to limits of functions

Specified, unspecified, and undefined quantities

1 Specified quantity

It is the quantity which has determined result , for example $\frac{2}{3}$ is a specified quantity .

2 Unspecified quantity

It is the quantity which has no determined result , for example $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \times \infty$, $\infty - \infty$, $(\text{zero})^{\text{zero}}$, $(\infty)^{\text{zero}}$ and $(1)^{\infty}$

3 Undefined quantity

It is the quantity which is meaningless , for example $\frac{4}{0}$ is undefined quantity .

Properties of ∞ , $-\infty$

$$(1) \infty \pm a = \infty , -\infty \pm a = -\infty$$

$$(2) \infty \times a = \begin{cases} \infty & , a > 0 \\ -\infty & , a < 0 \\ \text{unspecified} & , a = 0 \end{cases}$$

The concept of the limit of a function at a point

Illustrated example

If we want to find the value of the function $f : f(x) = \frac{x^4 - 1}{x - 1}$ at $x = 1$, we find its value = $\frac{0}{0}$ which is unspecified value .

That mean we can not determine the value of the function at $x = 1$, so we go to study the approaching of $f(x)$ to a specified quantity , when x approach to the number 1 , that by one of the following two methods :

① Evaluation of the limit numerically:

x approaches to 1 from left ----->						<----- x approaches to 1 from right					
x	0.5	0.6	0.7	0.8	0.9		1.1	1.2	1.3	1.4	1.5
f(x)	1.5	1.6	1.7	1.8	1.9		2.1	2.2	2.3	2.4	2.5
f(x) approaches to 1 from left ----->						<----- f(x) approaches to 1 from right					

We find that:

When x approaches to the number 1 from the right side " $x \rightarrow 1^+$ " and is read as : (x tends to one from the right) , then f(x) approaches to the number 2 and the number 2 is called the right limit of the function and it is written mathematically as : $\lim_{x \rightarrow 1^+} f(x) = 2$ or $f(1^+) = 2$ and is read as the limit of the function when (x $\rightarrow 1^+$) equals 2

When x approaches to the number 1 from the left side " $x \rightarrow 1^-$ " and is read as : (x tends to one from the left) , then f(x) approaches to the number 2 and the number 2 is called the right limit of the function and it is written mathematically as : $\lim_{x \rightarrow 1^-} f(x) = 2$ or $f(1^-) = 2$ and is read as the limit of the function when (x $\rightarrow 1^-$) equals 2

Definition

If the value of the function f approaches to a unique value L when x approaches to a from the two sides right and left , then the limit of f(x) equals L and is written symbolically :

$$\lim_{x \rightarrow a} f(x) = L$$

i.e. If $f(a^+) = f(a^-) = L$, then $\lim_{x \rightarrow a} f(x) = L$

In the previous example : $f(1^+) = f(1^-) = 2$, then $\lim_{x \rightarrow 1} f(x) = 2$

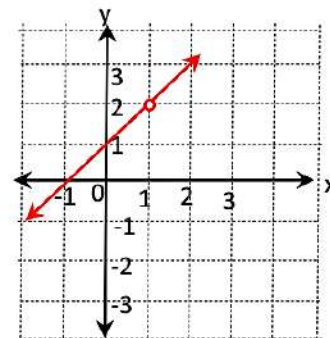
② Evaluation of the limit graphically:

$\therefore f(x) = \frac{x^2 - 1}{x - 1}$ is undefined at $x = 1$

$$\therefore f(x) = \frac{(x-1)(x+1)}{(x-1)} = (x+1), x \neq 1$$

It is represented by a straight line with an open dot at the point whose x-coordinate = 1 as in the opposite figure, and from the figure we notice that when x tends to 1 (from the right and the left), then $f(x)$ tends to 2

$$\therefore \lim_{x \rightarrow 1} f(x) = 2$$

**Remarks:**

[1] At finding $\lim_{x \rightarrow a} f(x)$, it is not necessary that the function be defined at $x = a$, it should be defined only in an interval on the left of a and another interval on the right of a .

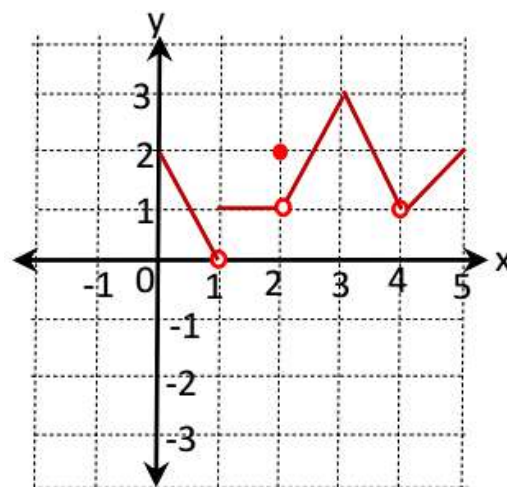
[2] If $f(a^+) \neq f(a^-)$, then $\lim_{x \rightarrow a} f(x)$ does not exist.

(1) In the opposite figure, we find that :

First : at $x = 0$: $\lim_{x \rightarrow 0^+} f(x) = 1$

But $\lim_{x \rightarrow 0^-} f(x)$ and $\lim_{x \rightarrow 0} f(x)$ don't exist

(Because the function is undefined at the left of $x = 0$)



Second : at $x = 1$: $\lim_{x \rightarrow 1^-} f(x) = 0$, $\lim_{x \rightarrow 1^+} f(x) = 1$

$\therefore \lim_{x \rightarrow 1} f(x)$ does not exist (because the right limit $\neq$ the left limit)

Notice that : although f is defined at $x = 1$, $f(1) = 1$, the limit does not exist.

Third : at $x = 2$: $\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2} f(x) = 1$

Notice that : it is not necessary that the value of the function equals the value of the limit where $f(2) = 2$

Fourth : at $x = 3$: $\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3} f(x) = f(3) = 3$

Fifth : at $x = 4$: $\lim_{x \rightarrow 4^-} f(x) = \lim_{x \rightarrow 4^+} f(x) = \lim_{x \rightarrow 4} f(x) = 1$

Notice that : $f(4)$ is undefined i.e. Although the function is undefined , the limit exists

Sixth : at $x = 5$: $\lim_{x \rightarrow 5^-} f(x) = 2$, $\lim_{x \rightarrow 5^+} f(x)$ and $\lim_{x \rightarrow 5} f(x)$ don't exist

(Because the function is undefined on the right of $x = 5$)

Examples

[1] Estimate the existence of a limit of each of the following functions when $x = 2$

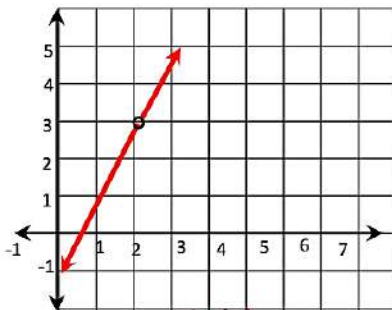


Fig (1)

$$F(2^+) =$$

$$F(2^-) =$$

$$\lim_{x \rightarrow 2} F(x) =$$

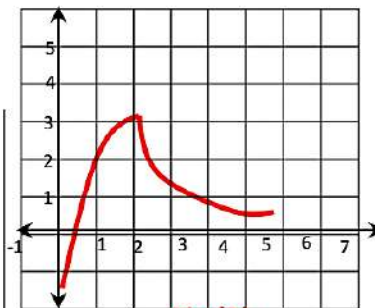


Fig (2)

$$F(2^+) =$$

$$F(2^-) =$$

$$\lim_{x \rightarrow 2} F(x) =$$

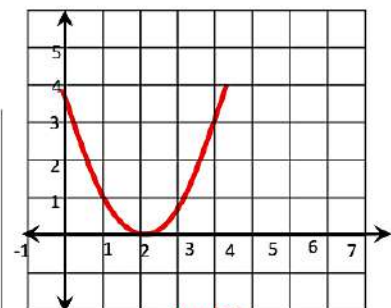


Fig (3)

$$F(2^+) =$$

$$F(2^-) =$$

$$\lim_{x \rightarrow 2} F(x) =$$

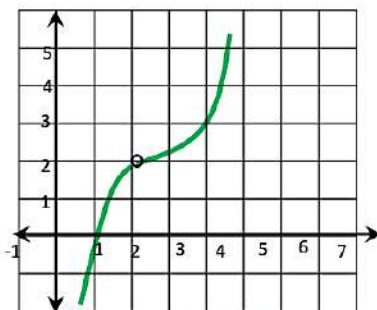


Fig (4)

$$F(2^+) =$$

$$F(2^-) =$$

$$\lim_{x \rightarrow 2} F(x) =$$

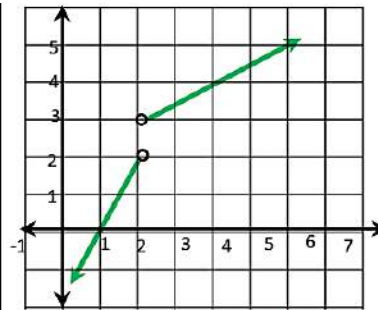


Fig (5)

$$F(2^+) =$$

$$F(2^-) =$$

$$\lim_{x \rightarrow 2} F(x) =$$

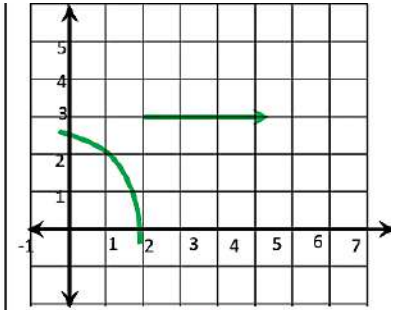


Fig (6)

$$F(2^+) =$$

$$F(2^-) =$$

$$\lim_{x \rightarrow 2} F(x) =$$

[2] From the opposite figures:

$$F(0^+) =$$

$$F(0^-) =$$

$$\lim_{x \rightarrow 0} F(x) =$$

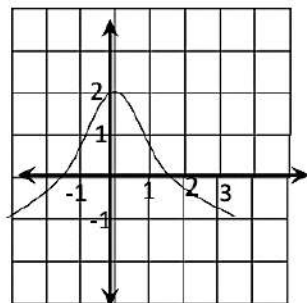


Fig (1)

$$F(1^+) =$$

$$F(1^-) =$$

$$\lim_{x \rightarrow 1} F(x) =$$

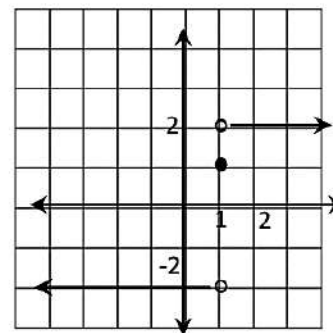


Fig (2)

$$F(2^+) =$$

$$F(2^-) =$$

$$\lim_{x \rightarrow 2} F(x) =$$

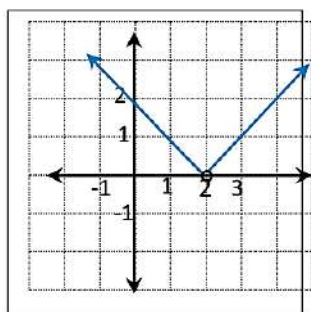


Fig (3)

$$F(-1^+) =$$

$$F(-1^-) =$$

$$\lim_{x \rightarrow -1} F(x) =$$

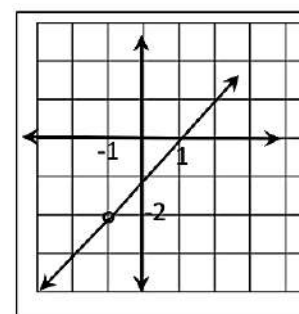


Fig (4)

$$F(0^+) =$$

$$F(0^-) =$$

$$\lim_{x \rightarrow 0} F(x) =$$

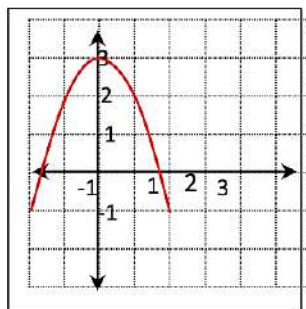


Fig (5)

$$F(-1^+) =$$

$$F(-1^-) =$$

$$\lim_{x \rightarrow -1} F(x) =$$

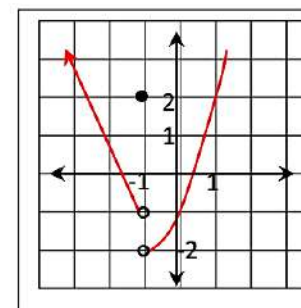


Fig (6)

$$F(1^+) =$$

$$F(1^-) =$$

$$\lim_{x \rightarrow 1} F(x) =$$

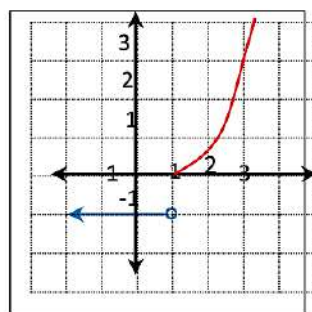


Fig (7)

$$F(-1^+) =$$

$$F(-1^-) =$$

$$\lim_{x \rightarrow -1} F(x) =$$

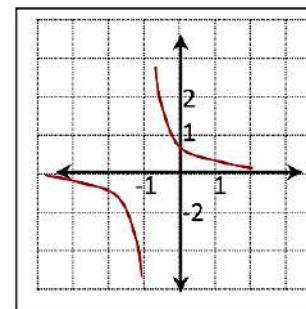
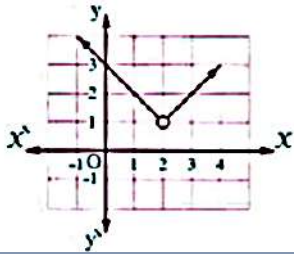
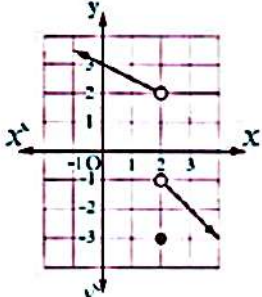
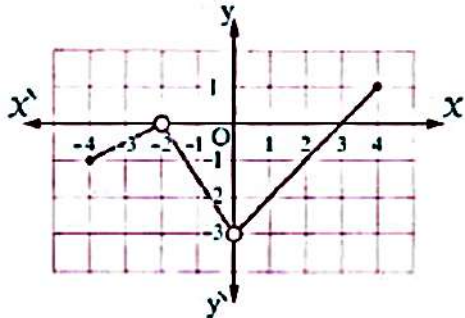


Fig (8)

Exercises (1)

Choose the correct answer:

1)	All the following are unspecified quantities except (a) zero $\div$ zero (b) $\infty - \infty$ (c) $\infty + \infty$ (d) $\infty \div \infty$	
2)	In the opposite figure : $\lim_{x \rightarrow 2} f(x) = \dots\dots\dots$ (a) 1 (b) -1 (c) does not exist. (d) 2	
3)	In the opposite figure : $\lim_{x \rightarrow 2} f(x) = \dots\dots\dots$ (a) -3 (b) 2 (c) -1 (d) does not exist.	
4)	 From the opposite figure : First : $\lim_{x \rightarrow -2} f(x) = \dots\dots\dots$ (a) zero (b) -3 (c) -2 (d) does not exist. Second : $\lim_{x \rightarrow 0} f(x) = \dots\dots\dots$ (a) zero (b) -2 (c) -3 (d) does not exist. Third : $\lim_{x \rightarrow -4} f(x) = \dots\dots\dots$ (a) zero (b) -4 (c) -1 (d) does not exist. Fourth : $\lim_{x \rightarrow 4} f(x) = \dots\dots\dots$ (a) zero (b) 4 (c) 1 (d) does not exist.	

5)

Using the opposite figure :

First : $\lim_{x \rightarrow 1^+} f(x) = \dots\dots\dots$

- (a) zero (b) -2
(c) 1 (d) does not exist.

Second : $\lim_{x \rightarrow 1^-} f(x) = \dots\dots\dots$

- (a) zero (b) -2 (c) 1 (d) does not exist.

Third : $\lim_{x \rightarrow 1} f(x) = \dots\dots\dots$

- (a) zero (b) -2 (c) 1 (d) does not exist.

Fourth : $\lim_{x \rightarrow -1^+} f(x) = \dots\dots\dots$

- (a) zero (b) -1 (c) -2 (d) does not exist.

Fifth : $\lim_{x \rightarrow -1^-} f(x) = \dots\dots\dots$

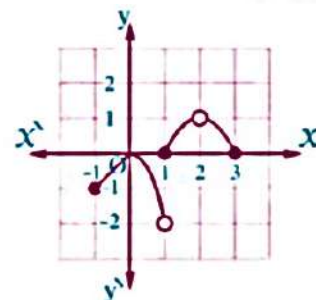
- (a) zero (b) -1 (c) -2 (d) does not exist.

Sixth : $\lim_{x \rightarrow 2} f(x) = \dots\dots\dots$

- (a) zero (b) 1 (c) 2 (d) does not exist.

Seventh : $\lim_{x \rightarrow 0} f(x) = \dots\dots\dots$

- (a) zero (b) -1 (c) -2 (d) does not exist.



6)

Using the opposite figure :

First : $\lim_{x \rightarrow -1^+} f(x) = \dots\dots\dots$

- (a) zero (b) -1
(c) 4 (d) does not exist.

Second : $\lim_{x \rightarrow -1^-} f(x) = \dots\dots\dots$

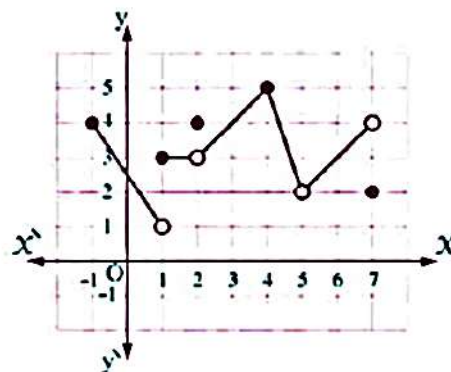
- (a) zero (b) -1 (c) 4 (d) does not exist.

Third : $\lim_{x \rightarrow -1} f(x) = \dots\dots\dots$

- (a) zero (b) -1 (c) 4 (d) does not exist.

Fourth : $f(2) = \dots\dots\dots$

- (a) zero (b) 3 (c) 4 (d) undefined.



Lesson (2): Finding the limit of a function algebraically

Direct substitution:

Theorem ①: If $f(x)$ is a polynomial, $a \in \mathbb{R}$, then $\lim_{x \rightarrow a} f(x) = f(a)$
i.e (Direct substitution)

Examples

[1] Find each of the following limits:

(a) $\lim_{x \rightarrow 2} (x^2 - 3x + 5)$

(b) $\lim_{x \rightarrow 3} (-5)$

(c) $\lim_{x \rightarrow 3} (2x - 5)$

(d) $\lim_{x \rightarrow -2} (3x^2 + x - 4)$

(e) $\lim_{x \rightarrow 2} \frac{3}{x-2}$

(f) $\lim_{x \rightarrow -5} \frac{x+5}{3x}$

Theorem ②: If f, g are two real functions in x , $\lim_{x \rightarrow a} f(x) = n$, $\lim_{x \rightarrow a} g(x) = m$ where :
 $L, m \in \mathbb{R}$, then :

(1) $\lim_{x \rightarrow a} [f(x) \pm g(x)] = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x) = n \pm m$

(2) $\lim_{x \rightarrow a} [f(x) \times g(x)] = \lim_{x \rightarrow a} f(x) \times \lim_{x \rightarrow a} g(x) = n \times m$

(3) $\lim_{x \rightarrow a} k f(x) = k \lim_{x \rightarrow a} f(x) = k n$ where k is constant

(4) $\lim_{x \rightarrow a} [f(x) \div g(x)] = \lim_{x \rightarrow a} f(x) \div \lim_{x \rightarrow a} g(x) = n \div m, m \neq 0$

(5) $\lim_{x \rightarrow a} [f(x)]^n = \left[\lim_{x \rightarrow a} f(x) \right]^n = L^n, n \in \mathbb{Z}^+$

Using factorization:

[2] Find each of the following limits:

$$(1) \lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1}$$

$$(2) \lim_{x \rightarrow 2} \frac{x^2 - x - 2}{2x - 4}$$

$$(3) \lim_{x \rightarrow -3} \frac{x^2 + 8x + 15}{x^2 - 4x - 21}$$

$$(4) \lim_{x \rightarrow 1} \frac{x^3 - 1}{x - 1}$$

$$(5) \lim_{x \rightarrow \sqrt{3}} \frac{x^2 - 3}{\sqrt{3}x - 3}$$

$$(6) \lim_{x \rightarrow -2} \frac{x^4 - 3x^2 - 4}{2x^2 - 8}$$

$$(7) \lim_{x \rightarrow -2} \frac{x^2 - 2x - 8}{x^3 + 8}$$

$$(8) \lim_{x \rightarrow -0} \frac{(x-1)^2 - 1}{4x}$$

$$(9) \lim_{x \rightarrow -0} \frac{(3x-2)^2 - 4}{5x}$$

$$(10) \lim_{x \rightarrow \frac{1}{2}} \frac{4x^2 - 1}{2x^2 + 5x - 3}$$

$$(11) \lim_{x \rightarrow \frac{3}{2}} \frac{8x^3 - 27}{2x^2 + 3x - 9}$$

$$(12) \lim_{x \rightarrow 3} \frac{\frac{1}{x} - \frac{1}{3}}{x - 3}$$

Finding limits by using long division method

[3] Find each of the following limits:

$$(1) \lim_{x \rightarrow 4} \frac{x^3 - 15x - 4}{x - 4}$$

$$(2) \lim_{x \rightarrow 2} \frac{3x^3 - 5x^2 - 3x + 2}{x^3 - 8}$$

Sol.

$$(1) \lim_{x \rightarrow 4} \frac{x^3 - 15x - 4}{x - 4} = \frac{0}{0} = \text{unspecified value}$$

$$\therefore \lim_{x \rightarrow 4} \frac{(x-4)(x^2 + 4x + 1)}{(x-4)} = \lim_{x \rightarrow 4} (x^2 + 4x + 1) = 33$$

1	0	-15	-4	4
0	4	16	4	
1	4	1	0	

$$(3) \lim_{x \rightarrow 1} \frac{x^3 + x^2 - 10x + 8}{x^3 + x^2 - 17x + 15}$$

$$(4) \lim_{x \rightarrow 1} \frac{5x^{-3} + 2x^{-1} - 7}{3x^{-2} + x^{-3} - 4}$$

Finding limits by using the conjugate

[3] Find each of the following limits:

$$(1) \lim_{x \rightarrow 4} \frac{\sqrt{x-3}-1}{x-4}$$

$$(2) \lim_{x \rightarrow 5} \frac{\sqrt{x-1}-2}{x-5}$$

$$(3) \lim_{x \rightarrow -1} \frac{x+1}{\sqrt{x+5}-2}$$

$$(4) \lim_{x \rightarrow 5} \frac{x^2-5x}{\sqrt{x+4}-3}$$

$$(5) \lim_{x \rightarrow 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{2x}$$

$$(6) \lim_{x \rightarrow 3} \frac{\sqrt{x+1}-2}{\sqrt{x-2}-1}$$

Exercises (2)

Choose the correct answer:

1)	$\lim_{x \rightarrow \frac{1}{2}} (10) = \dots\dots\dots$ (a) 5 (b) 20 (c) 10 (d) $10\frac{1}{2}$
2)	$\lim_{x \rightarrow 2} (3 a^2) = \dots\dots\dots$ (a) 3 (b) 6 (c) $3 a^2$ (d) 12
3)	$\lim_{x \rightarrow \frac{\pi}{2}} (2 x - \sin x) = \dots\dots\dots$ (a) π (b) $\pi + 1$ (c) $\pi - 1$ (d) $2 \pi - 1$
4)	$\lim_{x \rightarrow 3} \frac{2 x - 6}{7 x - 21} = \dots\dots\dots$ (a) $\frac{2}{3}$ (b) $\frac{2}{7}$ (c) $\frac{3}{7}$ (d) 3
5)	$\lim_{x \rightarrow 0} \frac{x^2 - x}{x} = \dots\dots\dots$ (a) zero. (b) -1 (c) does not exist. (d) 1
6)	$\lim_{x \rightarrow 3} \frac{x^2 - 7 x + 12}{x - 3} = \dots\dots\dots$ (a) 1 (b) -1 (c) 7 (d) -2
7)	$\lim_{x \rightarrow 3} \frac{x^2 - x - 6}{x^2 + x - 12} = \dots\dots\dots$ (a) $\frac{5}{7}$ (b) $\frac{1}{7}$ (c) -1 (d) -5

8)	$\lim_{x \rightarrow 0} \frac{3x + 2x^{-1}}{x + 4x^{-1}} = \dots\dots\dots$ (a) 2 (b) 4 (c) $\frac{1}{2}$ (d) $\frac{1}{4}$
9)	$\lim_{x \rightarrow \sqrt{5}} \frac{x^4 - x^2 - 20}{x - \sqrt{5}} = \dots\dots\dots$ (a) 9 (b) $2\sqrt{5}$ (c) $9\sqrt{5}$ (d) $18\sqrt{5}$
10)	$\lim_{x \rightarrow 0} \frac{\sqrt{x+1} - 1}{x} = \dots\dots\dots$ (a) zero (b) $\sqrt{2}$ (c) $\frac{1}{2}$ (d) has no existence.
11)	$\lim_{x \rightarrow 9} \frac{\sqrt{2} - \sqrt{x-7}}{x-9} = \dots\dots\dots$ (a) $2\sqrt{2}$ (b) $\frac{\sqrt{2}}{4}$ (c) $-\frac{\sqrt{2}}{4}$ (d) $-2\sqrt{2}$
12)	$\lim_{x \rightarrow 2} \frac{(x-3)^2 - 1}{\sqrt{x+2} - 2} = \dots\dots\dots$ (a) -6 (b) -8 (c) -2 (d) does not exist.
13)	$\lim_{x \rightarrow 1} \frac{\sqrt{2x-1} - 1}{\sqrt{3x+1} - 2} = \dots\dots\dots$ (a) 1 (b) $\frac{2}{3}$ (c) $\frac{1}{2}$ (d) $\frac{4}{3}$
14)	$\lim_{x \rightarrow 2} \frac{x^3 - 7x + 6}{3x^2 - 8x + 4} = \dots\dots\dots$ (a) $\frac{5}{4}$ (b) $\frac{3}{2}$ (c) $\frac{4}{5}$ (d) $\frac{1}{3}$

15)	$\lim_{x \rightarrow 0} \frac{7+2x}{\cos x} = \dots\dots\dots$ (a) 7 (b) 8 (c) 9 (d) 1
16)	$\lim_{x \rightarrow \frac{\pi}{4}} \frac{\tan x}{x} = \dots\dots\dots$ (a) 0 (b) 1 (c) $\frac{4}{\pi}$ (d) does not exist.
17)	If $\lim_{x \rightarrow m} \frac{2x^2 - x - 3}{4x^2 - 9} = \frac{5}{12}$, then m = $\dots\dots\dots$ (a) $\frac{3}{2}$ (b) $\frac{-3}{2}$ (c) $\frac{2}{3}$ (d) $\frac{-2}{3}$
18)	If $\lim_{x \rightarrow 2} \frac{x^2 - 4a}{x - 2}$ exists, then a = $\dots\dots\dots$ (a) -1 (b) 1 (c) 2 (d) 4

Lesson (3): Theorem (4) " the law"

Theorem:

$$\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = n a^{n-1}, \text{ for every } n \in \mathbb{R} - \{0\}$$

Corollaries:

$$(1) \lim_{x \rightarrow a} \frac{x^n - a^n}{x^m - a^m} = \frac{n}{m} a^{n-m} \quad (2) \lim_{x \rightarrow 0} \frac{(x+a)^n - a^n}{x} = n a^{n-1}$$

[1] Find each of the following limits:

$$(1) \lim_{x \rightarrow -2} \frac{x^5 - 32}{x^2 - 4}$$

$$(2) \lim_{x \rightarrow a} \frac{x^8 - a^8}{x - a}$$

$$(3) \lim_{x \rightarrow -3} \frac{x^5 + 243}{x + 3}$$

$$(4) \lim_{x \rightarrow -1} \frac{x^9 + 1}{x^4 - 1}$$

$$(5) \lim_{x \rightarrow -2} \frac{x^4 - 16}{x^5 + 32}$$

$$(6) \lim_{x \rightarrow 0} \frac{(x+5)^4 - 625}{x}$$

$$(7) \lim_{x \rightarrow 16} \frac{\sqrt[4]{x^5} - 32}{\sqrt{x^3} - 64}$$

$$(8) \lim_{x \rightarrow 16} \frac{\sqrt[4]{x^7} - 128}{x - 16}$$

$$(9) \lim_{x \rightarrow 7} \frac{\sqrt[5]{x+25} - 2}{x - 7}$$

$$(10) \lim_{x \rightarrow \frac{3}{2}} \frac{16x^5 - 81x}{2x^3 - 3x^2}$$

$$(11) \lim_{x \rightarrow \frac{\pi}{6}} \frac{\sin^3 x - \frac{1}{8}}{\sin x - \frac{1}{2}}$$

$$(12) \lim_{h \rightarrow 0} \frac{(1+4h)^8 - 1}{h}$$

(13) $\lim_{h \rightarrow 0} \frac{(a+2h)^6 - a^6}{5h}$

(14) $\lim_{x \rightarrow 0} \frac{(3-2x)^4 - 81}{3x}$

(15) $\lim_{x \rightarrow 1} \frac{\sqrt[5]{4x-3} - 1}{x-1}$

Exercises (3)

Choose the correct answer:

1)	$\lim_{x \rightarrow a} \frac{x^n - a^n}{x^m - a^m} = \dots\dots\dots$ (a) $\frac{m}{n}$ (b) $\frac{m}{n} (a)^{m-n}$ (c) $\frac{n}{m} (a)^{m-n}$ (d) $\frac{n}{m} (a)^{n-m}$
2)	$\lim_{y \rightarrow 2} \frac{y^5 - 32}{y - 2} = \dots\dots\dots$ (a) $31 y^4$ (b) 32×2^4 (c) 64 (d) 5×2^4
3)	$\lim_{x \rightarrow 1} \frac{x^{10} - 1}{x - 1} = \dots\dots\dots$ (a) zero. (b) 1 (c) 9 (d) 10

4)	$\text{Lim}_{x \rightarrow 2} \frac{x^5 - 32}{x^3 - 8} = \dots\dots\dots$ (a) 4 (b) $\frac{5}{3}$ (c) zero (d) $6\frac{2}{3}$
5)	$\text{Lim}_{x \rightarrow \frac{1}{2}} \frac{4x^2 - 1}{32x^5 - 1} = \dots\dots\dots$ (a) $\frac{2}{5}$ (b) $\frac{5}{2}$ (c) $\frac{2}{9}$ (d) $\frac{1}{8}$
6)	$\text{Lim}_{x \rightarrow 16} \frac{\sqrt[4]{x^5 - 32}}{x - 16} = \dots\dots\dots$ (a) 5 (b) $\frac{5}{2}$ (c) $\frac{5}{4}$ (d) $\frac{5}{8}$
7)	$\text{Lim}_{h \rightarrow 0} \frac{(2 - 3h)^7 - 128}{4h} = \dots\dots\dots$ (a) 336 (b) - 336 (c) 448 (d) - 448
8)	$\text{Lim}_{h \rightarrow 0} \frac{(3h - 1)^5 + 1}{5h} = \dots\dots\dots$ (a) - 3 (b) $\frac{3}{5}$ (c) 3 (d) 5
9)	$\text{Lim}_{x \rightarrow 1} \frac{x^{\frac{13}{2}} - x^{\frac{1}{2}}}{x^{\frac{7}{2}} - x^{\frac{1}{2}}} = \dots\dots\dots$ (a) $\frac{13}{7}$ (b) 1 (c) 2 (d) x
10)	$\text{Lim}_{x \rightarrow 0} \frac{(x + 2)^5 - 32}{x} = \dots\dots\dots$ (a) 25 (b) 64 (c) 80 (d) 100
11)	$\text{Lim}_{x \rightarrow 2} \frac{(x + 1)^4 - 81}{x - 2} = \dots\dots\dots$ (a) 18 (b) 81 (c) - 108 (d) 108

12)	$\lim_{x \rightarrow 1} \frac{1 - \sqrt[n]{x}}{1 - \sqrt[m]{x}} = \dots\dots\dots$ (a) 1 (b) $\frac{n}{m}$ (c) -1 (d) $\frac{m}{n}$
13)	$\lim_{x \rightarrow 9} \frac{3 - \sqrt{x}}{27 - \sqrt{x^3}} = \dots\dots\dots$ (a) $\frac{1}{9}$ (b) $\frac{1}{27}$ (c) 3 (d) $-\frac{1}{27}$
14)	$\lim_{x \rightarrow 2} \frac{x^5 - 32}{x^2 + 3x - 10} = \dots\dots\dots$ (a) 80 (b) $\frac{80}{7}$ (c) $\frac{7}{80}$ (d) $\frac{1}{80}$
15)	$\text{If } \lim_{x \rightarrow 2} \frac{(x)^n - (2)^n}{x - 2} = 32, \text{ then } n = \dots\dots\dots$ (a) 3 (b) 4 (c) 9 (d) 12
16)	$\text{If } \lim_{x \rightarrow a} \frac{x^8 - a^8}{x^6 - a^6} = 48, \text{ then } a = \dots\dots\dots$ (a) 4 (b) 6 (c) ± 4 (d) ± 6
17)	$\lim_{x \rightarrow 0} \frac{(1+x)^{\frac{5}{2}} - 1}{x} = \dots\dots\dots$ (a) $\frac{5}{2}$ (b) zero. (c) $\frac{2}{5}$ (d) does not exist.
18)	$\lim_{x \rightarrow 0} \frac{(x+2)^6 - 64}{x^2 + 16x} = \dots\dots\dots$ (a) 6 (b) 12 (c) 16 (d) 64

Lesson (4): Limit of the function at infinity

Theorem: $\lim_{x \rightarrow \infty} \frac{1}{x} = 0$

Corollaries: $\lim_{x \rightarrow \infty} \frac{a}{x^n} = 0$, where $n \in \mathbb{R}^+$, a is constant

Note that:

(1) $\lim_{x \rightarrow \infty} c = c$

(2) If n is positive integer, the $\lim_{x \rightarrow \infty} x^n = \infty$

(3) The theorem above applied on the sum, difference, product and quotient of the two functions as $x \rightarrow a$

(4) When finding $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)}$ where $f(x)$ and $g(x)$ are polynomials, then:

* If the degree of $f(x)$ = the degree of $g(x)$, then $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \text{real no.}$

* If the degree of $f(x)$ > the degree of $g(x)$, then $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \pm \infty$

* If the degree of $f(x)$ < the degree of $g(x)$, then $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \text{zero}$

[1] Find each of the following limits:

(1) $\lim_{x \rightarrow \infty} \left(\frac{1}{x} + 3 \right)$

(2) $\lim_{x \rightarrow \infty} \left(4 - \frac{3}{x^2} \right)$

(3) $\lim_{x \rightarrow \infty} \left(\frac{5}{x} + 2 \right)$

(4) $\lim_{x \rightarrow \infty} \left(\frac{2}{x^2} + 5 \right)$

(5) $\lim_{x \rightarrow \infty} (4x - x^3 + 5)$

(6) $\lim_{x \rightarrow \infty} (x^3 + 7x^2 + 2)$

$$(7) \lim_{x \rightarrow \infty} \frac{2x-3}{3x^2+1}$$

$$(8) \lim_{x \rightarrow \infty} \frac{2x^2-3}{3x^2+1}$$

$$(9) \lim_{x \rightarrow \infty} \frac{2x^3-3}{3x^2+1}$$

$$(10) \lim_{x \rightarrow \infty} \frac{5x^2-3x+1}{2x}$$

$$(11) \lim_{x \rightarrow \infty} \frac{4x^3-5x}{4x^4+3x^2-2}$$

$$(12) \lim_{x \rightarrow \infty} \frac{-6x^2+1}{3x^2+x-2}$$

$$(13) \lim_{x \rightarrow \infty} \frac{x^3-2}{|x|^3+1}$$

$$(14) \lim_{x \rightarrow \infty} (x - \sqrt{x^2 + 4})$$

$$(15) \lim_{x \rightarrow \infty} \frac{x-3}{\sqrt{4x^2+25}}$$

Exercises (4)

Choose the correct answer:

1)	$\lim_{x \rightarrow \infty} \left(\frac{3}{x^2} - 2 \right) = \dots\dots\dots$ (a) 3 (b) 2 (c) -3 (d) -2
2)	$\lim_{x \rightarrow \infty} \frac{3x}{4x+5} = \dots\dots\dots$ (a) ∞ (b) $\frac{3}{4}$ (c) $\frac{1}{5}$ (d) zero
3)	$\lim_{x \rightarrow \infty} \frac{2x^2+1}{x^2+1} = \dots\dots\dots$ (a) zero. (b) does not exist. (c) ∞ (d) 2
4)	$\lim_{x \rightarrow \infty} \frac{x^2+5}{6} = \dots\dots\dots$ (a) zero. (b) $\frac{5}{6}$ (c) 1 (d) ∞
5)	$\lim_{x \rightarrow \infty} (3x^{-5} + 4x^{-2} + 5) = \dots\dots\dots$ (a) 12 (b) ∞ (c) 5 (d) zero
6)	$\lim_{x \rightarrow \infty} \frac{3x^{-3} + 4x^{-2} - 2}{7x^{-3} - x^{-2} + 6} = \dots\dots\dots$ (a) ∞ (b) zero (c) $\frac{3}{7}$ (d) $-\frac{1}{3}$
7)	$\lim_{x \rightarrow \infty} \frac{x^7 - 2x^3}{2x^4 - 3x^2 - 1} = \dots\dots\dots$ (a) zero (b) 3 (c) ∞ (d) $\frac{1}{2}$

8)	$\lim_{x \rightarrow \infty} \sqrt{\frac{3+2x}{4x-1}} = \dots\dots\dots$ (a) $\frac{1}{2}$ (b) $\frac{3}{4}$ (c) $\frac{1}{\sqrt{2}}$ (d) $\frac{\sqrt{3}}{2}$
9)	$\lim_{x \rightarrow \infty} \frac{x^3+5}{x(2x^2+3)} = \dots\dots\dots$ (a) $\frac{5}{8}$ (b) 1 (c) $\frac{1}{2}$ (d) $\frac{5}{3}$
10)	$\lim_{x \rightarrow \infty} \frac{1}{x} \sqrt{8+9x^2} = \dots\dots\dots$ (a) $2\sqrt{2}$ (b) 3 (c) $-2\sqrt{2}$ (d) -3
11)	$\lim_{x \rightarrow \infty} \frac{\sqrt[3]{64x^3+7x-2}}{3x+2} = \dots\dots\dots$ (a) 4 (b) 3 (c) $\frac{2}{3}$ (d) $\frac{4}{3}$
12)	$\lim_{x \rightarrow \infty} \frac{\sqrt{9x^2+3}}{\sqrt[3]{1-8x^3}} = \dots\dots\dots$ (a) $\frac{-2}{3}$ (b) $\frac{-9}{8}$ (c) $\frac{3}{2}$ (d) $\frac{-3}{2}$
13)	$\lim_{x \rightarrow \infty} \left(\frac{3x^2+2x+1}{x^2-3x+2} \right)^4 = \dots\dots\dots$ (a) 3 (b) 9 (c) 27 (d) 81
14)	If $\lim_{x \rightarrow \infty} \frac{a^2x+7}{2x-5} = 8$, then $a = \dots\dots\dots$ where $a \in \mathbb{R}$ (a) 2 (b) zero. (c) ± 4 (d) ± 8

Lesson (5): Limits of trigonometric functions

Theorem: If x is the measure of an angle in radians, then:

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \quad , \quad \lim_{x \rightarrow 0} \frac{\tan x}{x} = 1$$

Corollaries:

$$(1) \lim_{x \rightarrow 0} \frac{\sin ax}{x} = a \quad , \quad \lim_{x \rightarrow 0} \frac{\tan ax}{x} = a \quad (2) \lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0$$

Note that: (1) $\sin^2 x + \cos^2 x = 1$

$$(2) \cos 2x = \begin{cases} \cos^2 x - \sin^2 x \\ 2\cos^2 x - 1 \\ 1 - 2\sin^2 x \end{cases}$$

$$(3) \sin^2 x = (\sin x)^2 \neq \sin x^2$$

[1] Find each of the following limits:

$$(1) \lim_{x \rightarrow 0} \frac{\sin 3x}{x}$$

$$(2) \lim_{x \rightarrow 0} \frac{\tan 2x}{7x}$$

$$(3) \lim_{x \rightarrow 0} \frac{\sin 5x \cos 2x}{x}$$

$$(4) \lim_{x \rightarrow 0} \frac{1 - \cos x}{\tan x}$$

$$(5) \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2}$$

$$(6) \lim_{x \rightarrow 0} 6x^2 \csc 2x \cot x$$

$$(7) \lim_{x \rightarrow 0} \frac{1 - \cos x}{\sin 2x}$$

$$(8) \lim_{x \rightarrow 0} \frac{x^2 - x + \sin x}{2x}$$

$$(9) \lim_{x \rightarrow \frac{\pi}{2}} \frac{\cos x}{\pi - 2x}$$

$$(10) \lim_{x \rightarrow 0} \frac{x + x \cos x}{\sin x \cos x}$$

$$(11) \lim_{x \rightarrow 0} \frac{\sin^2 3x}{5x^2}$$

$$(12) \lim_{x \rightarrow 0} \frac{\sin 3x}{5x^3 - 4x}$$

$$(13) \lim_{x \rightarrow 1} \frac{\sin(x-1)}{x^2 + x - 2}$$

$$(14) \lim_{x \rightarrow 0} \frac{x \sin 2x + \sin^2 2x}{\tan^2 3x + x^2}$$

$$(15) \lim_{x \rightarrow 1} \frac{\sin \pi x}{1-x}$$

Exercises (5)

Choose the correct answer:

1)	$\lim_{x \rightarrow 0} \tan x = \dots\dots\dots$ (a) zero (b) -1 (c) $\frac{\pi}{2}$ (d) π	
2)	$\lim_{x \rightarrow \frac{\pi}{2}} \sin 2x = \dots\dots\dots$ (a) zero (b) -1 (c) 1 (d) $\frac{\pi}{2}$	
3)	$\lim_{x \rightarrow 0} \frac{2x}{\sin 3x} = \dots\dots\dots$ (a) $\frac{2}{3}$ (b) $\frac{3}{2}$ (c) 6 (d) has no existence.	
4)	$\lim_{(2x-3) \rightarrow 0} \frac{\tan(2x-3)}{2x-3} = \dots\dots\dots$ (a) zero (b) -1 (c) 1 (d) does not exist.	
5)	$\lim_{h \rightarrow 0} \frac{\sin 3h^2}{4h^2} = \dots\dots\dots$ (a) zero (b) 1 (c) $\frac{4}{3}$ (d) $\frac{3}{4}$	
6)	$\lim_{x \rightarrow 0} \frac{3x^5}{\tan 4x^5} = \dots\dots\dots$ (a) $\frac{3}{4}$ (b) $\frac{4}{3}$ (c) 1 (d) zero.	
7)	$\lim_{x \rightarrow 0} \frac{1 - \cos x}{3x} = \dots\dots\dots$ (a) zero. (b) 1 (c) -1 (d) does not exist.	
8)	$\lim_{x \rightarrow 0} \frac{4+5x}{\cos 3x} = \dots\dots\dots$ (a) -4 (b) 4 (c) 3 (d) -3	

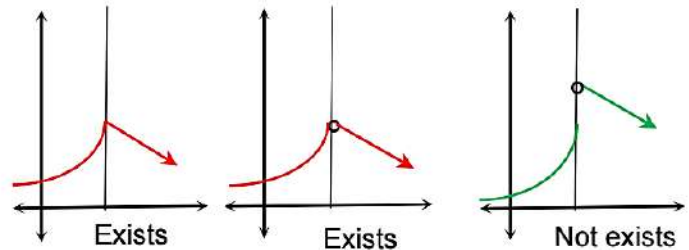
9)	$\lim_{x \rightarrow 0} \frac{\sin \frac{x}{3}}{2x} = \dots\dots\dots$ (a) $\frac{1}{3}$ (b) $\frac{1}{2}$ (c) $\frac{1}{6}$ (d) $\frac{1}{12}$
10)	$\lim_{x \rightarrow 0} \frac{2x + \sin 3x}{\tan 5x} = \dots\dots\dots$ (a) 5 (b) $\frac{6}{5}$ (c) 1 (d) zero.
11)	$\lim_{x \rightarrow 0} \frac{2x + \sin 3x}{5x + \tan 2x} = \dots\dots\dots$ (a) 1 (b) $\frac{5}{7}$ (c) $\frac{7}{5}$ (d) - 1
12)	$\lim_{x \rightarrow 0} \frac{\sin \frac{1}{2}x}{\sin \frac{3}{4}x} = \dots\dots\dots$ (a) $\frac{1}{6}$ (b) $\frac{3}{8}$ (c) $\frac{1}{2}$ (d) $\frac{2}{3}$
13)	$\lim_{x \rightarrow 0} \frac{\tan^2 2x}{x \sin 3x} = \dots\dots\dots$ (a) $\frac{4}{9}$ (b) $\frac{1}{2}$ (c) $\frac{2}{3}$ (d) $\frac{4}{3}$
14)	$\lim_{x \rightarrow 0} \frac{x^2 + \sin^2 2x}{x \tan 2x} = \dots\dots\dots$ (a) $\frac{2}{5}$ (b) $\frac{5}{2}$ (c) 1 (d) 5
15)	$\lim_{x \rightarrow 0} 3x \csc 2x = \dots\dots\dots$ (a) 6 (b) $\frac{3}{2}$ (c) $\frac{2}{3}$ (d) has no existence.

16)	$\text{Lim}_{x \rightarrow 0} \frac{\sin 2x \tan 3x}{4x^2} = \dots\dots\dots$ (a) $\frac{1}{2}$ (b) $\frac{3}{4}$ (c) $\frac{3}{2}$ (d) 6
17)	$\text{Lim}_{x \rightarrow \pi} \frac{\sin x}{\pi - x} = \dots\dots\dots$ (a) 1 (b) π^2 (c) π (d) $-\pi$
18)	$\text{Lim}_{x \rightarrow 0} \frac{\sin x}{x} = \dots\dots\dots$, where x is in degrees. (a) 1 (b) $\frac{\pi}{180}$ (c) $\frac{180}{\pi}$ (d) π
19)	$\text{Lim}_{x \rightarrow 1} \frac{\sin (x-1)}{x^2-1} = \dots\dots\dots$ (a) $\frac{1}{2}$ (b) 1 (c) -1 (d) zero.
20)	$\text{Lim}_{x \rightarrow 3} \frac{\tan (x-3)}{x^2-x-6} = \dots\dots\dots$ (a) $\frac{1}{5}$ (b) $\frac{8}{3}$ (c) 3 (d) -3

Lesson (6): Existence of the limit of a piecewise function

Limits at a point "a"

Theorem: $\lim_{x \rightarrow a} f(x)$ if: $f(a^+) = f(a^-)$



[1] Estimate the existence of limit of each of the following function when $x = 2$

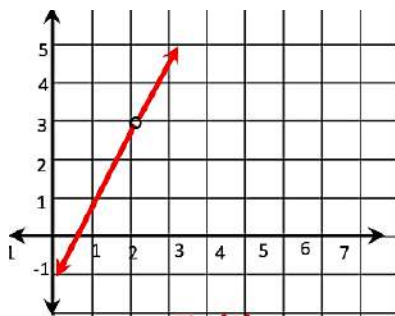


Fig (1)

$F(2^+) =$ $F(2^-) =$
 $\lim_{x \rightarrow 2} F(x) =$
 $F(2) =$
 $\therefore F(x)$ is

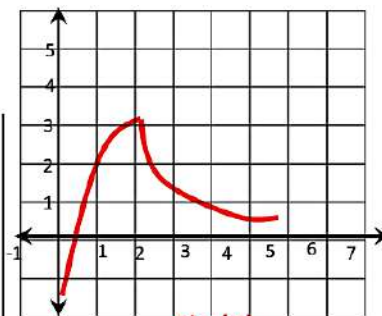


Fig (2)

$F(2^+) =$ $F(2^-) =$
 $\lim_{x \rightarrow 2} F(x) =$
 $F(2) =$
 $\therefore F(x)$ is

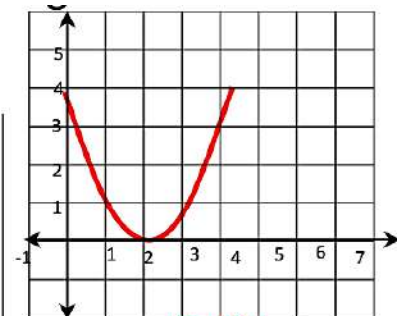


Fig (3)

$F(2^+) =$ $F(2^-) =$
 $\lim_{x \rightarrow 2} F(x) =$
 $F(2) =$
 $\therefore F(x)$ is

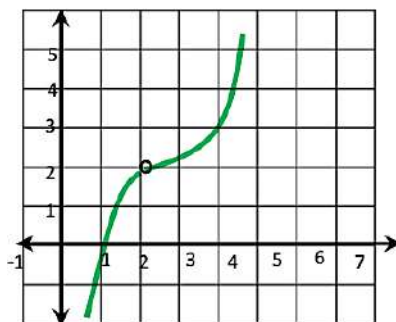


Fig (4)

$F(2^+) =$ $F(2^-) =$
 $\lim_{x \rightarrow 2} F(x) =$
 $F(2) =$
 $\therefore F(x)$ is

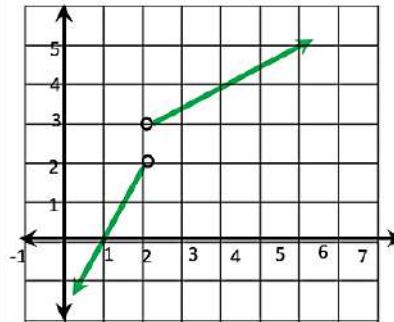


Fig (5)

$F(2^+) =$ $F(2^-) =$
 $\lim_{x \rightarrow 2} F(x) =$
 $F(2) =$
 $\therefore F(x)$ is

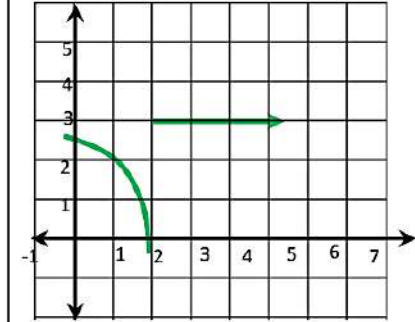


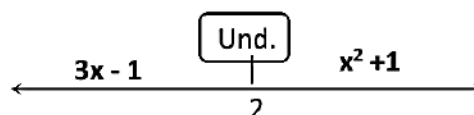
Fig (6)

$F(2^+) =$ $F(2^-) =$
 $\lim_{x \rightarrow 2} F(x) =$
 $F(2) =$
 $\therefore F(x)$ is

(1) $F(x) = \begin{cases} x^2 + 1 & , x > 2 \\ 3x - 1 & , x < 2 \end{cases}$. Discuss the limit existence at $x = 2$

Sol.

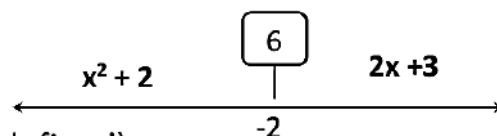
$$\begin{aligned} \# F(2^+) &= \lim_{x \rightarrow 2^+} x^2 + 1 = 2^2 + 1 = 5 \\ F(2^-) &= \lim_{x \rightarrow 2^-} 3x - 1 = 3(2) - 1 = 5 \\ \therefore F(2^+) &= F(2^-) = 5 \quad \therefore \lim_{x \rightarrow 2} F(x) = 5 \end{aligned}$$



(2) $f(x) = \begin{cases} x^2 + 2 & , x \leq -2 \\ 2x + 3 & , x > -2 \end{cases}$. Discuss the limit existence at $x = -2$

Sol.

$$\begin{aligned} \# F(-2^+) &= \lim_{x \rightarrow -2^+} 2x + 3 = 2(-2) + 3 = -1 \\ F(-2^-) &= \lim_{x \rightarrow -2^-} x^2 + 2 = (-2)^2 + 2 = 6 \\ \therefore F(-2^+) &\neq F(-2^-) \quad \therefore \lim_{x \rightarrow -2} F(x) \text{ does not exist (undefined)} \end{aligned}$$



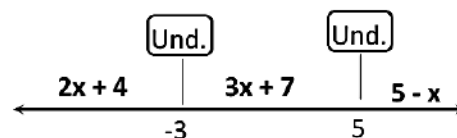
(3) $(x) = \begin{cases} 2x + 4 & x < -3 \\ 3x + 7 & -3 < x < 5 \\ 5 - x & x > 5 \end{cases}$. Discuss the limit existence of:

(i) $\lim_{x \rightarrow -3} F(x)$

(ii) $\lim_{x \rightarrow 5} F(x)$

Sol.

$$\begin{aligned} \# F(-3^-) &= 2(-3) + 4 = -2, \quad F(-3^+) = 3(-3) + 7 = -2 \\ \therefore F(-3^-) &= F(-3^+) \quad \therefore \lim_{x \rightarrow -3} F(x) = -2 \text{ (exist)} \\ \# F(5^-) &= 3(5) + 7 = 22, \quad F(5^+) = 5 - 5 = 0 \\ \therefore F(5^-) &\neq F(5^+) \quad \therefore \lim_{x \rightarrow 5} F(x) \text{ does not exist (undefined)} \end{aligned}$$



(4) $(x) = \begin{cases} \frac{7x - \sin 5x}{x + 3 \tan x} & , x > 0 \\ \frac{x-2}{x-4} & , x < 0 \end{cases}$. Discuss the limit existence at $x = 0$

(5) Discuss the existence of $\lim_{x \rightarrow 1} \frac{x-1}{|x-1|}$

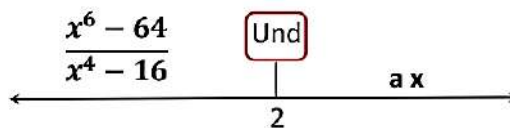
(6) $f F(x) = \begin{cases} \frac{x^6 - 64}{x^4 - 16} & , x < 2 \\ ax & , x > 2 \end{cases}$ Where $\lim_{x \rightarrow 2} F(x)$ exists, find the value of a

Sol.

$\therefore \lim_{x \rightarrow 2} F(x)$ exists $\therefore F(2^+) = F(2^-) \therefore F(2^+) = 2a$

$F(2^-) = \lim_{x \rightarrow 2^-} \frac{x^6 - 64}{x^4 - 16} = \lim_{x \rightarrow 2^-} \frac{x^6 - 2^6}{x^4 - 2^4} = \frac{6}{4} (2)^{6-4} = 6$

$\therefore 2a = 6 \therefore a = 3$



(7) If $F(x) = \begin{cases} a + \cos 2x & , x < 0 \\ \frac{\sin 2x}{ax} & , x > 0 \end{cases}$, $\lim_{x \rightarrow 0} F(x)$ exists, find the value of a

(8) If $F(x) = \begin{cases} ax^2 + b & , x \geq 0 \\ \frac{a - a \cos x}{x^2} & , x < 0 \end{cases}$ And $\lim_{x \rightarrow 0} F(x) = 2$, find the value of a, b

Exercises (6)

Choose the correct answer:

1)	If $f(x) = \begin{cases} -2 & , & x > 0 \\ 1 & , & x < 0 \end{cases}$, then : First : $\lim_{x \rightarrow -3} f(x) = \dots\dots\dots$ (a) - 2 (b) 1 (c) - 1 (d) does not exist. Second : $\lim_{x \rightarrow 4} f(x) = \dots\dots\dots$ (a) - 2 (b) 1 (c) - 1 (d) does not exist. Third : $\lim_{x \rightarrow 0} f(x) = \dots\dots\dots$ (a) - 2 (b) 1 (c) - 1 (d) does not exist.
2)	If $f(x) = \begin{cases} x^2 - 1 & , & x > 2 \\ 3x + 1 & , & x \leq 2 \end{cases}$, then $\lim_{x \rightarrow 3} f(x) = \dots\dots\dots$ (a) does not exist. (b) 3 (c) 8 (d) 7
3)	If $f(x) = \begin{cases} 3x - 1 & , & x \neq 2 \\ 6 & , & x = 2 \end{cases}$, then $\lim_{x \rightarrow 2} f(x) = \dots\dots\dots$ (a) - 5 (b) 5 (c) 6 (d) does not exist.
4)	If $f(x) = \begin{cases} 3 - x & , & x < 1 \\ 4 & , & x = 1 \\ x^2 + 1 & , & x > 1 \end{cases}$, then $\lim_{x \rightarrow 1} f(x) = \dots\dots\dots$ (a) 4 (b) 2 (c) 1 (d) does not exist.

5)

If $f(x) = \begin{cases} \frac{\sin x}{x} & , -\frac{\pi}{2} < x < 0 \\ \cos x & , 0 < x < \frac{\pi}{2} \end{cases}$, then $\lim_{x \rightarrow 0} f(x) = \dots\dots\dots$

(a) $\frac{2}{\pi}$

(b) $\frac{\pi}{2}$

(c) 1

(d) 0

6)

If $f(x) = \begin{cases} \frac{\sin(x-1)}{x-1} & , x < 1 \\ \tan \frac{\pi x}{4} & , 1 < x < 2 \end{cases}$, then $\lim_{x \rightarrow 1} f(x) = \dots\dots\dots$

(a) 1

(b) $\frac{1}{4}$

(c) $\frac{\pi}{4}$

(d) does not exist.

7)

If $f(x) = \begin{cases} x+1 & \text{at } x > a \\ 3x-1 & \text{at } x < a \end{cases}$, $\lim_{x \rightarrow a} f(x)$ exist, then $a = \dots\dots\dots$

(a) zero

(b) 1

(c) 2

(d) 3

8)

If $f(x) = \begin{cases} a + \cos x & , x < 0 \\ \frac{\tan 2x}{a x} & , 0 < x < \frac{\pi}{4} \end{cases}$ and $\lim_{x \rightarrow 0} f(x)$ exists, then $a = \dots\dots\dots$

(a) zero or 1

(b) 1 or -2

(c) 2 or 3

(d) 1 or 2

9)

If $f(x) = \begin{cases} \frac{\cot x}{\pi - 2x} & , 0 < x < \frac{\pi}{2} \\ \sin(\pi - x) & , x > \frac{\pi}{2} \end{cases}$, then $\lim_{x \rightarrow \frac{\pi}{2}} f(x) = \dots\dots\dots$

(a) 2

(b) $\frac{\pi}{2}$

(c) π

(d) does not exist.

10)

If $f(x) = \begin{cases} \frac{x^6 - 64}{x^4 - 16} & , x < 2 \\ a x & , x > 2 \end{cases}$

where $\lim_{x \rightarrow 2} f(x)$ exists, then the value of $a = \dots\dots\dots$

(a) 1

(b) 2

(c) 3

(d) 6

Lesson (7): Continuity

First: Continuity of a function at a point.

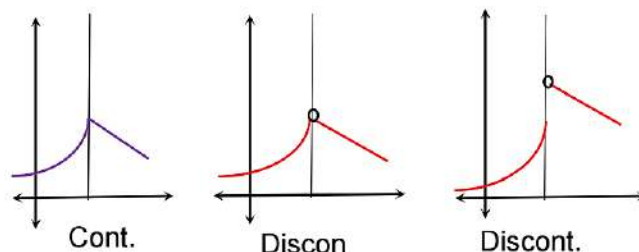
Definition: The function $F(x)$ is said to

be continuous at $x = a$ if:

(i) $F(x)$ is defined at $x = a$ ($a \in$ the domain of F)

(ii) $\lim_{x \rightarrow a} F(x)$ exists

(iii) $\lim_{x \rightarrow a} F(x) = F(a)$



Remarks:

(1) If any of these three conditions is not satisfied, then the function is said to be discontinuous at $x = a$.

(2) If $F(x)$ is continuous at $x = a$ then: $\lim_{x \rightarrow a} F(x) = F(a)$

(3) If $F(x)$ is discontinuous at $x = a$ and $\lim_{x \rightarrow a} F(x)$ exists then we can redefine $F(x)$ to be continuous at $x = a$.

(4) If $F(x)$ is discontinuous at $x = a$ and $\lim_{x \rightarrow a} F(x)$ does not exist at $x = a$, then we cannot redefine $F(x)$ to be continuous at $x = a$.

Examples

(1) Discuss the continuity of the function: $F(x) = |x - 1| + 2$ at $x = 1$

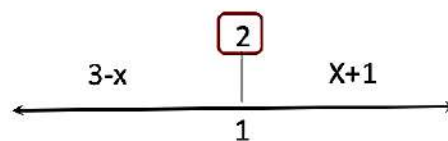
Sol.

$$\# F(x) = \begin{cases} x - 1 + 2 & , x \geq 1 \\ -x + 1 + 2 & , x < 1 \end{cases} = \begin{cases} x + 1 & , x \geq 1 \\ 3 - x & , x < 1 \end{cases}$$

(i) $F(1) = 2 \quad \therefore F(x)$ is defined at $x = 1$

(ii) $\lim_{x \rightarrow 1} F(x) = \begin{cases} F(1^+) = 2 \\ F(1^-) = 2 \end{cases} \therefore \lim_{x \rightarrow 1} F(x) = 2$

(iii) $\therefore F(1) = \lim_{x \rightarrow 1} F(x) = 2 \quad \therefore F(x)$ is continuous at $x = 1$



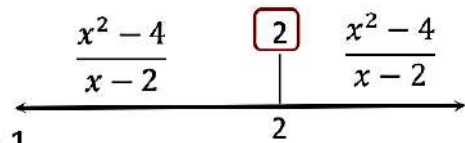
(2) Discuss the continuity of the function: $F(x) = \begin{cases} \frac{x^2-4}{x-2} & x \neq 2 \\ 2 & x = 2 \end{cases}$ at $x = 2$

Sol.

(i) $F(2) = 2 \quad \therefore F(x)$ is defined at $x = 2$

$$(ii) \lim_{x \rightarrow 2} F(x) = \lim_{x \rightarrow 2} \frac{x^2-4}{x-2} = \lim_{x \rightarrow 2} \frac{(x-2)(x+2)}{(x-2)} = 4$$

(iii) $\therefore F(2) \neq \lim_{x \rightarrow 2} F(x) \quad \therefore F(x)$ is discontinuous at $x = 2$



(3) Discuss the continuity of the: $F(x) = \begin{cases} \frac{2x+\sin x}{x} & x \neq 0 \\ \frac{1}{3} & x = 0 \end{cases}$ at $x = 0$

(4) If $F(x) = \begin{cases} \frac{5x+\tan 3x}{\sin 4x} & x \neq 0 \\ a & x = 0 \end{cases}$ is continuous at $x = 0$, find the value of a

Sol.

$\therefore F(x)$ is continuous at $x = 0 \quad \therefore F(0^+) = F(0^-) = F(0)$

$$\therefore a = \lim_{x \rightarrow 0} \frac{5x+\tan 3x}{\sin 4x} = \lim_{x \rightarrow 0} \frac{5 + \frac{\tan 3x}{x}}{\frac{\sin 4x}{x}} = \frac{5+3}{4} = 2$$

(5) If $F(x) = \begin{cases} x^2 + ax - 2 & x < 2 \\ 4 & x = 2 \\ 5a + bx & x > 2 \end{cases}$ is continuous at $x = 2$, find the value of a and b

(6) Redefine $F(x) = \frac{2x^2 - x - 3}{4x^2 - 9}$ such that $F(x)$ becomes continuous at $x = \frac{3}{2}$

Sol.

$$\lim_{x \rightarrow \frac{3}{2}} \frac{2x^2 - x - 3}{4x^2 - 9} = \lim_{x \rightarrow \frac{3}{2}} \frac{(2x - 3)(x + 1)}{(2x - 3)(2x + 3)} = \frac{\frac{3}{2} + 1}{3 + 3} = \frac{5}{12}$$

$$\therefore F(x) \text{ is continuous at } x = \frac{3}{2} \quad \therefore F\left(\frac{3}{2}\right) = \lim_{x \rightarrow \frac{3}{2}} F(x) = \frac{5}{12}$$

$$\therefore \text{The new rule for continuous function is: } F(x) = \begin{cases} \frac{2x^2 - x - 3}{4x^2 - 9} & x \neq \frac{3}{2} \\ \frac{5}{12} & x = \frac{3}{2} \end{cases}$$

Exercises (7)

Choose the correct answer:

1)	The function $f(x)$ is continuous at $x = a$ if (a) $f(a)$ exist. (b) $f(a) = f(a^+) = f(a^-)$ (c) $f(x)$ has no limit at $x \rightarrow a$ (d) a and c together.	
2)	If the function $f : f(x) = \begin{cases} \frac{x^2 - 1}{x - 1} & , x \neq 1 \\ 2a & , x = 1 \end{cases}$ is continuous at $x = 1$, then $a = \dots\dots\dots$ (a) zero. (b) 1 (c) 2 (d) 4	
3)	If $f : f(x) = \begin{cases} ax^2 - 3 & , x \neq 2 \\ 2a & , x = 2 \end{cases}$ is continuous at $x = 2$, then $a = \dots\dots\dots$ (a) $\frac{1}{2}$ (b) $\frac{2}{3}$ (c) $\frac{3}{2}$ (d) 6	
4)	If $f : f(x) = \begin{cases} x^2 - 2x & \text{at } x \geq 1 \\ kx - 3 & \text{at } x < 1 \end{cases}$ is continuous at $x = 1$, then $k = \dots\dots\dots$ (a) zero (b) 1 (c) 2 (d) 3	
5)	If the function f is continuous at $x = 2$ where $f(x) = \begin{cases} ax^2 + 5 & , \text{ at } x \leq 2 \\ 9 - bx & , \text{ at } x > 2 \end{cases}$, then $2a + b = \dots\dots\dots$ (a) 7 (b) 14 (c) 2 (d) - 2	

6)	If $f : f(x) = \begin{cases} \frac{(x+2)^6 - 64}{2x} & , \quad x \neq 0 \\ k & , \quad x = 0 \end{cases}$ is continuous at $x = 0$, then $k = \dots\dots\dots$ (a) 96 (b) 192 (c) 384 (d) 92
7)	If the function $f : f(x) = \begin{cases} \frac{\sin 2x + \tan x}{1 + \sin 4x} & , \quad x \in]0, \frac{\pi}{3}[- \{\frac{\pi}{4}\} \\ k & , \quad x = \frac{\pi}{4} \end{cases}$ is continuous at $x = \frac{\pi}{4}$, then $k = \dots\dots\dots$ (a) $\frac{1}{2}$ (b) $\frac{3}{5}$ (c) 3 (d) 2
8)	If $f : f(x) = \begin{cases} \frac{\sin ax}{2x} & , \quad x \neq 0 \\ 5 & , \quad x = 0 \end{cases}$ is continuous on $\mathbb{R}$, then $a = \dots\dots\dots$ (a) 5 (b) 10 (c) 2 (d) zero.
9)	The function $f : f(x) = \frac{1}{\sqrt[3]{x+2}}$ is continuous for each $x \in \dots\dots\dots$ (a) $\mathbb{R}$ (b) $\mathbb{R} - \{-2\}$ (c) $[-2, \infty[$ (d) $] -2, \infty[$
10)	The function $f : f(x) = \frac{x}{ x-1 -3}$ is continuous on $\dots\dots\dots$ (a) $\mathbb{R} - \{3\}$ (b) $\mathbb{R} - \{-3, 3\}$ (c) $\mathbb{R} - \{-2, 4\}$ (d) $\mathbb{R}$

Unit four

Trigonometry

Lesson ① The sine rule

Lesson ② The cosine rule

Lesson ③ Solution of the triangle

Lesson (1): The sine rule

In any triangle, the lengths of the sides are proportional to the sines of the opposite angles i.e $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

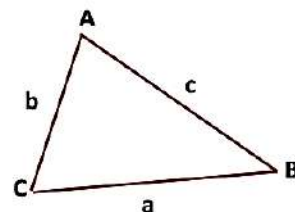
Proof: Area of triangle = $\frac{1}{2} b c \sin A = \frac{1}{2} a c \sin B = \frac{1}{2} a b \sin C$ (Multiplying by 2)

$$\therefore b c \sin A = a c \sin B = a b \sin C \text{ (dividing by } a b c \text{)}$$

$$\therefore \frac{b c \sin A}{a b c} = \frac{a c \sin B}{a b c} = \frac{a b \sin C}{a b c}$$

$$\therefore \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

$$\therefore \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



Using the sine rule to find the length of a side of a triangle

[1] Find the length of the longest side in the triangle ABC in which $m(\angle A) = 54^\circ 33'$, $m(\angle B) = 49^\circ 22'$, $a = 124.5$ cm.

Sol.

$\therefore$ The longest side is opposite to the greatest angle

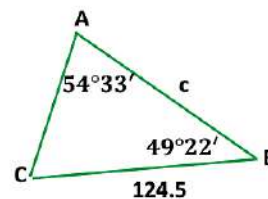
$$m(\angle C) = 180^\circ - [m(\angle A) + m(\angle B)] =$$

$$180^\circ - [54^\circ 33' + 49^\circ 22'] = 76^\circ 5'$$

$\therefore$ The longest side is c

$$\therefore \frac{a}{\sin A} = \frac{c}{\sin C} \quad \therefore \frac{124.5}{\sin 54^\circ 33'} = \frac{c}{\sin 76^\circ 5'}$$

$$\therefore c = \frac{124.5 \sin 76^\circ 5'}{\sin 54^\circ 33'} = 148.4 \text{ cm}$$



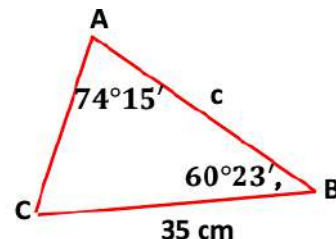
[2] In the triangle ABC in which $m(\angle A) = 74^\circ 15'$, $m(\angle B) = 60^\circ 23'$, $a = 35$ cm. Find a, b.

Sol.

$$m(\angle C) = 180^\circ - [74^\circ 15' + 60^\circ 23'] = 45^\circ 22'$$

$$\therefore \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad \therefore \frac{35}{\sin 74^\circ 15'} = \frac{b}{\sin 60^\circ 23'} = \frac{c}{\sin 45^\circ 22'}$$

$$\therefore b = 31.6 \text{ cm}, c = 25.9 \text{ cm.}$$



[3] In the triangle ABC in which $m(\angle A) = 110^\circ$, $m(\angle B) = 35^\circ$, $a = 436.2$ cm.
Find a, b, and the area of ΔABC .

Sol.

[4] ABCD is a parallelogram, $AB = 8$ cm, $m(\angle BAC) = 55^\circ$, $m(\angle ABD) = 20^\circ$. Find:
(i) The length of $\overline{BD}$, $\overline{AC}$
(ii) S. area of the parallelogram ABCD.

Sol.

Geometric Application of the Sine Rule:

In any triangle ABC: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2r$

Where r is the length of the radius of the circumcircle of ΔABC

[5] In ΔABC , if $a = 4$ cm, $m(\angle A) = 40^\circ$. Find the surface area of the circumcircle of ΔABC .

Sol.

$$\therefore \frac{a}{\sin A} = 2r \quad \therefore 2r = \frac{4}{\sin 40^\circ} \quad \therefore r = 3.1 \text{ cm}$$

$$\therefore \text{S. area of the circumcircle of } \Delta ABC = \pi r^2 = \pi (3.1)^2 = 30.2 \text{ cm}$$

[6] In ΔABC , if $m(\angle A) = 32^\circ$, $m(\angle B) = 126^\circ$ and its perimeter = 79 cm. Find:

i) The diameter of the circle passing through its vertices.

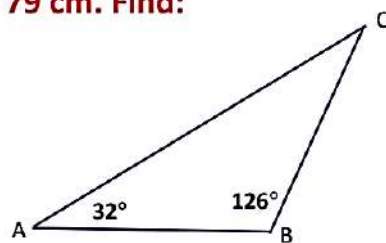
ii) The length of the smallest side.

Sol.

$$m(\angle C) = 180^\circ - (126^\circ + 32^\circ) = 22^\circ$$

$$\therefore \frac{\text{Perimeter of } \Delta ABC}{\sin A + \sin B + \sin C} = 2r \quad \therefore 2r = \frac{79}{1.7135} = 46.11 \text{ cm.}$$

$$\therefore \text{The diameter of the circle} = 2r = 46.11 \text{ cm.}$$



[7] Prove that the area of $\Delta ABC = \frac{abc}{4r}$ where r is the length of the radius passing through

its vertices, and if its area = $40\sqrt{3}$ cm², $b = 16$ cm, $c = 10$ cm, $r = \frac{14\sqrt{3}}{3}$ cm.

Find a and $m(\angle A)$.

Sol.

$$\therefore \text{Area of } \Delta ABC = \frac{1}{2} ab \sin C, \quad \therefore \frac{c}{\sin 22^\circ} = 2r \quad \therefore \sin C = \frac{c}{2r}$$

$$\therefore \text{Area of } \Delta ABC = \frac{1}{2} ab \times \frac{c}{2r} = \frac{abc}{4r}$$

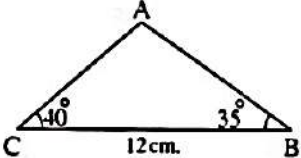
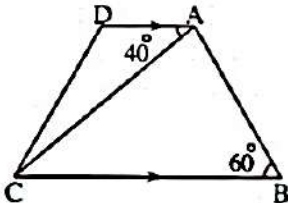
$$\therefore 40\sqrt{3} = \frac{a \times 16 \times 10}{4 \times \frac{14\sqrt{3}}{3}} \quad \therefore a = \frac{4 \times 14\sqrt{3} \times 40\sqrt{3}}{3 \times 16 \times 10} = 14 \text{ cm}$$

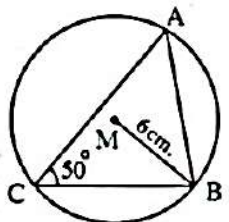
$$\therefore \frac{a}{\sin A} = 2r \quad \therefore \frac{14}{\sin A} = 2 \times \frac{14\sqrt{3}}{3} \quad \therefore 28\sqrt{3} \sin A = 42$$

$$\therefore \sin A = \frac{\sqrt{3}}{2} \quad \therefore m(\angle A) = 60^\circ \text{ or } 120^\circ$$

Exercises (1)

Choose the correct answer:

1)	In any triangle XYZ , $XY : YZ = \dots\dots\dots$ (a) $\sin X : \sin Y$ (b) $\sin Y : \sin Z$ (c) $\sin Z : \sin X$ (d) $\sin Z : \sin Y$	
2)	In $\triangle ABC$, if $m(\angle A) = 30^\circ$, $C = 15\sqrt{3}$ cm. , $m(\angle C) = 60^\circ$, then $a = \dots\dots\dots$ cm. (a) 30 (b) 45 (c) 15 (d) 60	
3)	DEF is a triangle in which $m(\angle D) = 80^\circ$ and $m(\angle E) = 60^\circ$, if $f = 12$ cm. , then $d = \dots\dots\dots$ cm. (a) $\frac{12 \sin 80^\circ}{\sin 40^\circ}$ (b) $\frac{12 \sin 80^\circ}{\sin 60^\circ}$ (c) $\frac{12 \sin 40^\circ}{\sin 80^\circ}$ (d) $\frac{12 \cos 80^\circ}{\cos 40^\circ}$	
4)	In $\triangle ABC$, if $a = 4$ cm. , $b = 7$ cm. , $m(\angle C) = 120^\circ$, then the area of the triangle = $\dots\dots\dots$ cm^2 . (a) $7\sqrt{3}$ (b) $14\sqrt{3}$ (c) 7 (d) 14	
5)	 XYZ is an equilateral triangle , the length of its side is $10\sqrt{3}$ cm. , then the length of the diameter of its circumcircle is $\dots\dots\dots$ cm. (a) 5 (b) 10 (c) 15 (d) 20	
6)	In the opposite figure : The length of $\overline{AB} \approx \dots\dots\dots$ cm. (a) 6 (b) 7 (c) 8 (d) 9	
7)	In the opposite figure : $\overline{AD} \parallel \overline{BC}$, $AB = 4$ cm. , $m(\angle DAC) = 40^\circ$, $m(\angle B) = 60^\circ$, then the length of $\overline{AC} \approx \dots\dots\dots$ cm. (a) 5 (b) 3 (c) 2 (d) 4	

8)	In the opposite figure : M is the centre of the circle BM = 6 cm. , then AB = cm. (a) $6 \sin 50^\circ$ (b) $12 \sin 50^\circ$ (c) $6 \cos 50^\circ$ (d) $12 \cos 50^\circ$	
9)	A circle with diameter of length 20 cm. , passes through the vertices of $\triangle ABC$ which is an acute-angled triangle in which $BC = 10$ cm. , then $m(\angle A) = \dots\dots\dots^\circ$ (a) 30 (b) 60 (c) 45 (d) 150	
10)	In triangle ABC , $m(\angle A) = 45^\circ$, the length of the radius of its circumcircle = 6 cm. , then $a = \dots\dots\dots$ cm. (a) 13 (b) $6\sqrt{2}$ (c) 12 (d) $\sqrt{2}$	
11)	If the perimeter of triangle ABC equals 15 cm. , $m(\angle A) = 53^\circ$, $m(\angle B) = 47^\circ$, then the length of $\overline{AB} = \dots\dots\dots$ cm. (a) 6 (b) 7 (c) 5 (d) 8	
12)	In triangle ABC , $a = 27$ cm. , $m(\angle B) = 82^\circ$, $m(\angle C) = 56^\circ$, then its surface area $\approx \dots\dots\dots \text{cm}^2$. (a) 540 (b) 447 (c) 350 (d) 400	
13)	In triangle ABC , $m(\angle A) : m(\angle B) : m(\angle C) = 2 : 3 : 4$, $AB = 12$ cm. , then the length of $\overline{AC} \approx \dots\dots\dots$ cm. (a) 10 (b) 11 (c) 16 (d) 18	
14)	 If r is the length of the radius of the circumcircle of the triangle XYZ , then $\frac{y}{2 \sin Y} = \dots\dots\dots$ (a) r (b) $2r$ (c) $\frac{1}{2}r$ (d) $4r$	

15)	In ΔABC , $\sin A = 2 \sin C$, $BC = 6$ cm. , then $AB = \dots\dots\dots$ cm. (a) 2 (b) 3 (c) 4 (d) 6
16)	If the radius length of circumcircle of ΔABC equals 3 cm. and $\sin A + \sin B + \sin C = 2$, then the perimeter of triangle $ABC = \dots\dots\dots$ cm. (a) 6 (b) 9 (c) 12 (d) 24
17)	In any triangle ABC , $\frac{\sin (A+B)}{\sin A + \sin B} = \dots\dots\dots$ (a) 1 (b) $\frac{c}{a+b}$ (c) $\frac{a}{b+c}$ (d) $\frac{b}{a+c}$
18)	In ΔABC , $\frac{a}{a+b} = \frac{\sin A}{\dots\dots\dots}$ (a) $\sin B$ (b) $\sin C$ (c) $\sin A + \sin B$ (d) $\sin A + \sin C$
19)	 In ΔXYZ , if $3 \sin X = 4 \sin Y = 2 \sin Z$, then $x : y : z = \dots\dots\dots$ (a) 2 : 3 : 4 (b) 6 : 4 : 3 (c) 3 : 4 : 6 (d) 4 : 3 : 6
20)	 ABC is a triangle in which $\frac{\sin A}{3} = \frac{2 \sin B}{5} = \frac{\sin C}{4}$, then $a : b : c = \dots\dots\dots$ (a) 6 : 5 : 8 (b) 8 : 5 : 6 (c) 7 : 2 : 4 (d) 3 : 5 : 4
21)	In ΔABC , $a - b = 4$ cm. , $\sin A = \frac{3}{2} \sin B$, then $a = \dots\dots\dots$ cm. (a) 4 (b) 6 (c) 8 (d) 12
22)	 In ΔABC , $\frac{2b}{\sin B} = \dots\dots\dots r$ (where r is the radius of its circumcircle) (a) 1 (b) 2 (c) 4 (d) 8

Lesson (2): The cosine rule

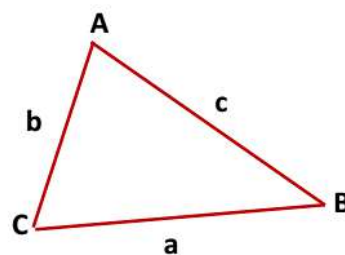
In any triangle ABC:

$$a^2 = b^2 + c^2 - 2 b c \cos A$$

$$b^2 = c^2 + a^2 - 2 c a \cos B$$

$$c^2 = a^2 + b^2 - 2 a b \cos C$$

$$\begin{aligned} \cos A &= \frac{b^2 + c^2 - a^2}{2 b c} \\ \cos B &= \frac{c^2 + a^2 - b^2}{2 c a} \\ \cos C &= \frac{a^2 + b^2 - c^2}{2 a b} \end{aligned}$$



These rules are used when two sides
and the included angle is known and
The required is to get the 3rd side

These rules are used when we have
3 sides and angles are required

Remarks:

(1) In ΔABC : $m(\angle A) + m(\angle B) + m(\angle C) = 180$

$$\therefore m(\angle A) + m(\angle B) = 180 - m(\angle C)$$

$$\therefore \cos(A + B) = -\cos C$$

(2) If ABCD is a cyclic quadrilateral, then:

$$\therefore m(\angle A) + m(\angle C) = 180$$

$$\therefore \cos A = -\cos C$$

Examples

[1] ΔABC in which $a = 7.12$ cm, $b = 6.43$ cm, $m(\angle C) = 75^\circ$. Find c .

Sol.

$$\therefore c^2 = a^2 + b^2 - 2 a b \cos C$$

$$\therefore c^2 = (7.12)^2 + (6.43)^2 - 2(7.12)(6.43) \cos 75^\circ$$

$$\therefore c^2 = 68.34$$

$$\therefore c = 8.3 \text{ cm.}$$

[2] $\triangle ABC$ in which $a = 45$ cm, $b = 75$ cm, $c = 105$ cm. Find measure of the greatest angle in $\triangle ABC$.

Sol.

$\therefore$ Side c is the longest side in $\triangle ABC$

$\therefore \angle C$ is the greatest angle

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{(45)^2 + (75)^2 - (105)^2}{2(45)(75)} = -\frac{1}{2}$$

$$\therefore m(\angle C) = 120^\circ$$

[3] $ABCD$ is a quadrilateral, $m(\angle A) = 60^\circ$, $AB = 5.6$ cm, $BC = 2.1$ cm, $AD = CD = 3.5$ cm. prove that $ABCD$ is cyclic quadrilateral.

Sol.

In $\triangle ABD$:

$$\begin{aligned}(BD)^2 &= (5.6)^2 + (3.5)^2 - 2(5.6)(3.5) \cos 60 \\ &= 24.01 \quad \therefore BD = 4.9 \text{ cm.}\end{aligned}$$

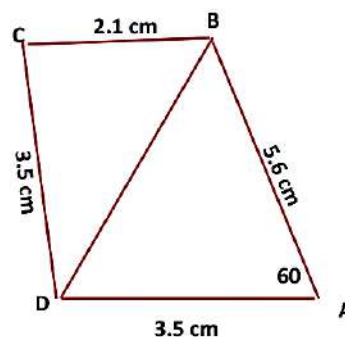
In $\triangle CBD$:

$$\cos C = \frac{(2.1)^2 + (3.5)^2 - (4.9)^2}{2(3.5)(4.9)} = -\frac{1}{2}$$

$$\therefore m(\angle C) = 120^\circ$$

$$\therefore m(\angle A) + m(\angle C) = 180^\circ$$

$\therefore ABCD$ is a cyclic quadrilateral.



[4] $ABCD$ is a quadrilateral in which $AB = 3$ cm, $AC = 8$ cm, $BC = 7$ cm, $CD = 5$ cm, $BD = 8$ cm. prove that the shape $ABCD$ is a cyclic quadrilateral.

Sol.

In $\triangle ABC$:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{(8)^2 + (3)^2 - (7)^2}{2(8)(3)} = \frac{1}{2}$$

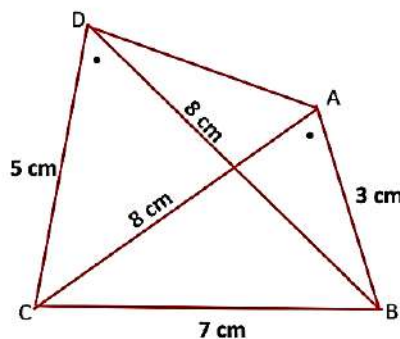
$$\therefore m(\angle A) = 60^\circ \quad (1)$$

In $\triangle BDC$:

$$\cos D = \frac{(CD)^2 + (DB)^2 - (BC)^2}{2(CD)(DB)} = \frac{(5)^2 + (8)^2 - (7)^2}{2(5)(8)} = \frac{1}{2}$$

$$\therefore m(\angle D) = 60^\circ \quad (2)$$

From (1), (2) $\therefore m(\angle BAC) = m(\angle BDC)$ and they are on the same base $\overline{BC}$ and on the same side of it. $\therefore$ The shape $ABCD$ is a cyclic quadrilateral.



[6] ABCD is a parallelogram, $AB = 5.1$ cm, $BC = 2.5$ cm, $m(\angle B) = 70^\circ$. Find the lengths of its diagonals and the angle between them.

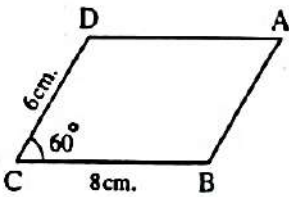
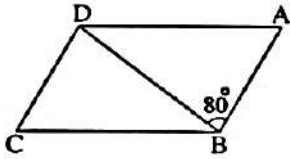
Sol.

Exercises (2)

Choose the correct answer:

1)	 In ΔXYZ, the expression $\frac{x^2 + y^2 - z^2}{2xy}$ equals (a) $\cos X$ (b) $\cos Y$ (c) $\cos Z$ (d) $\sin Z$	
2)	 In ΔXYZ, $y^2 + z^2 - x^2 = 2yz \times \dots\dots\dots$ (a) $\cos X$ (b) $\sin Z$ (c) $\cos Z$ (d) $\sin X$	
3)	In ΔABC, $\cos(A + B) = \dots\dots\dots$ (a) $\cos C$ (b) $-\cos C$ (c) $\sin C$ (d) $-\sin C$	
4)	If ABCD is a cyclic quadrilateral, then $\cos A + \cos C = \dots\dots\dots$ (a) 1 (b) zero. (c) $\frac{1}{2}$ (d) -1	

5)	In ΔXYZ , $2 X y \cos (X + Y) = \dots\dots\dots$ (a) $x^2 + y^2 - z^2$ (b) $y^2 + z^2 - x^2$ (c) $x^2 - z^2 - y^2$ (d) $z^2 - x^2 - y^2$
6)	In ΔLMN , $l = 5$ cm. , $m = 7$ cm. , $m(\angle N) = 60^\circ$, then $n = \dots\dots\dots$ cm. (to the nearest tenth) (a) 6.2 (b) 5 (c) 4.3 (d) 3.5
7)	In ΔXYZ , $x = 5$ cm. , $y = 3$ cm. , $m(\angle Z) = \frac{2}{3} \pi$, then $z = \dots\dots\dots$ (a) 7 (b) 8 (c) 9 (d) 4
8)	In ΔABC , if $m(\angle A) + m(\angle B) = 120^\circ$, $a = 2$ cm. , $b = 3$ cm. , then $c = \dots\dots\dots$ cm. (a) 4 (b) 3 (c) $\sqrt{7}$ (d) $\sqrt{5}$
9)	 In ΔXYZ , if $x = y$, then $\cos X = \dots\dots\dots$ (a) $\frac{2y^2}{z}$ (b) $\frac{z}{2y}$ (c) $\frac{z}{4x}$ (d) $\frac{y}{2x}$
10)	 In ΔABC , $\cos (A + B) = \dots\dots\dots$ (a) $\frac{a^2 + b^2 - c^2}{2ab}$ (b) $\frac{a^2 + c^2 - b^2}{2ab}$ (c) $\frac{b^2 + c^2 - a^2}{2bc}$ (d) $\frac{c^2 - a^2 - b^2}{2ab}$
11)	The measure of the greatest angle in triangle the lengths of its sides are 3 cm. , 5 cm. , 7 cm. equals $\dots\dots\dots^\circ$ (a) 110 (b) 150 (c) 100 (d) 120
12)	In ΔABC , if $\frac{\sin A}{\sin B} = 2 \cos C$, then $\dots\dots\dots$ (a) $b = c$ (b) $a = c$ (c) $a = b$ (d) $a = b = c$
13)	In ΔABC , $a^2 + b^2 - c^2 + \sqrt{3} ab = 0$, then $m(\angle C) = \dots\dots\dots^\circ$ (a) 30 (b) 150 (c) 60 (d) 120

14)	In triangle ABC , $c^2 = (a + b)^2 - ab$, then $m(\angle C)$° (a) 30 (b) 45 (c) 60 (d) 120	
15)	In $\triangle ABC$, $4 \sin A = 3 \sin B = 6 \sin C$, then $m(\angle C) =$ (to the nearest degree) (a) 89° (b) 29° (c) 57° (d) 82°	
16)	In $\triangle ABC$, $\frac{1}{2} \sin A = \frac{1}{3} \sin B = \frac{1}{4} \sin C$, then $\cos C =$ (a) $\frac{-2}{3}$ (b) $\frac{2}{3}$ (c) $\frac{-1}{4}$ (d) $\frac{1}{4}$	
17)	ABCD is a parallelogram in which $AB = 8$ cm. , $BC = 11$ cm. , $BD = 9$ cm. , then the length of $\overline{AC} =$ cm. (a) 9 (b) 10 (c) 11 (d) 17	
18)	ABCD is quadrilateral in which $AB = 22$ cm. , $BC = 25$ cm. , $DC = 18$ cm. , $m(\angle ADB) = 65^\circ$, $m(\angle DBA) = 50^\circ$, then $m(\angle CBD) =$ (a) $80^\circ 75'$ (b) $42^\circ 49' 19''$ (c) $44^\circ 28' 6''$ (d) $85^\circ 30'$	
19)	In the opposite figure : ABCD is a parallelogram , then $AC =$ cm. (a) $2\sqrt{13}$ (b) $2\sqrt{37}$ (c) $2\sqrt{17}$ (d) 148	
20)	In the opposite figure : ABCD is a parallelogram $m(\angle ABD) = 80^\circ$, $BD = 7$ cm. $AB = 5$ cm. , then the perimeter of parallelogram = to the nearest cm. (a) 25 (b) 26 (c) 29 (d) 30	

Lesson (3): Solution of the triangle

First case: two angles and side " Sin rule"

Solve the triangle ABC in which : $a = 8 \text{ cm}$, $m(\angle A) = 60^\circ$, $m(\angle B) = 40^\circ$

Sol.

Second case: two sides and including angle " Cos rule"

Solve the triangle ABC in which : $a = 11 \text{ cm}$, $b = 5 \text{ cm}$, $m(\angle C) = 20^\circ$

Sol.

Third case: three sides " Cos rule"Solve the triangle ABC in which : $a = 5 \text{ cm}$, $b = 2c = 8 \text{ cm}$ **Sol.****Exercises (3)****Choose the correct answer:**

1)	Solving the triangle means (a) to find the lengths of its sides. (b) to find the measures of its angles. (c) to find the relation between the lengths of its sides and the measures of its angles. (d) to find the lengths of its sides and the measures of its angles.	
2)	The perimeter of ΔABC , in which $b = 11 \text{ cm}$, $m(\angle A) = 67^\circ$, $m(\angle C) = 46^\circ$ equals (to the nearest cm.) (a) 22 (b) 38 (c) 31 (d) 27	
3)	 By solving the triangle ABC in which $a = 5 \text{ cm}$, $b = 7 \text{ cm}$, $m(\angle C) = 65^\circ$, then $c = \dots\dots\dots \text{ cm}$. (to the nearest tenth) (a) 4.4 (b) 2.1 (c) 6.7 (d) 8.2	

4)

By solving ΔABC in which $a = 2$ cm. , $b = 4\sqrt{2}$ cm. , $c = 2\sqrt{5}$ cm. , then

First : $\cos A = \dots\dots\dots$

(a) $\frac{3}{\sqrt{10}}$

(b) $\frac{4}{5}$

(c) $\frac{2}{\sqrt{10}}$

(d) $\frac{\sqrt{10}}{5}$

Second : $m(\angle C) = \dots\dots\dots$

(a) $32^\circ 18'$

(b) $27^\circ 43'$

(c) 135°

(d) 45°

5)

By solving ΔABC in which $a = 15$ cm. , $\cos B = \frac{1}{2}$, $\tan C = \frac{1}{\sqrt{3}}$

, then the perimeter of $\Delta ABC = \dots\dots\dots r$ "where r is the radius of the circumcircle"

(a) $(2 + \sqrt{2})$

(b) $(\frac{\sqrt{3}}{2} + 2)$

(c) $(3 + \sqrt{3})$

(d) 2